ON A LOCAL THEOREM FOR DENSITIES

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Abstract

Full Text

MATHEMATICS

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ON A LOCAL THEOREM FOR DENSITIES

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Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of independent random variables with the same distribution function $F(x)$ and finite variance. Without loss of generality, one may assume that their mathematical expectation is zero and their variance is equal to one. Denote by $F_n(x)$ the distribution function of the random variable

$$Z_n = \frac{\xi_1 + \xi_2 + \dots + \xi_n}{\sqrt{n}}.$$

It is known that $F_n(x)$, as a distribution function, can be represented uniquely in the form

$$F_n(x) = \int_{-\infty}^{x} P_n(x)\, dx + \psi_n(x),$$

where $P_n(x)$ is the corresponding probability density, and $\psi_n(x)$ is the singular part of the distribution.

In the present article the asymptotic behavior of the quantity

$$c_n = \int_{-\infty}^{+\infty} |P_n(x) - \varphi(x)|\, dx,$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Yu. V. Prokhorov ([1]) showed that condition

A. There exists an n_0 such that

$$\int_{-\infty}^{+\infty} P_{n_0}(x)\, dx > 0,$$

is necessary and sufficient for $c_n = o(1)$.

For the case when the summands have a finite moment of the third order, Yu. V. Prokhorov showed that under condition A

$$c_n = O\left(\sqrt{\frac{\ln n}{n}}\right).$$

In the case of existence of an absolute moment of order $2+\delta$, a similar result was established in work ([2]). Under more restrictive conditions, Yu. V. Prokhorov' s result that $c_n = o(1)$ was also obtained by V. Smith ([3]).

One of the authors of the present article, under the assumption that $P_{n_0}(x)$ is integrable to a power $\gamma > 1$, and assuming that ξ_i has a finite moment of third order α, succeeded in refining the asymptotic behavior of c_n, establishing that under condition A and these additional assumptions

$$c_n = \lambda \frac{|\alpha|}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right), \qquad (*)$$

where

$$\lambda = \frac{1}{3\sqrt{2\pi}}\left(1 + 4e^{-3/2}\right).$$

It turned out that the assumption of integrability to a power $\gamma > 1$ is superfluous. Namely:

Theorem 1. If condition A is satisfied and the summands ξ_i have a finite moment of third order a, then relation (*) holds.

Moreover, the following theorems are valid:

Theorem 2. If condition A is satisfied and the summands ξ_i have a finite moment of order $2+\delta$ $(0 < \delta \leqslant 1)$, then

$$c_n = O\left(n^{-\delta/2}\right).$$

Theorem 3. For any sequence $0 < \lambda_n \to 0$ there exists a distribution $F(x)$ satisfying condition A such that, as $n \to \infty$,

$$\frac{c_n}{\lambda_n} \to +\infty.$$

The proof of Theorems 1 and 2 requires many auxiliary propositions and computations. In essence, the method of proof of these theorems consists in a combination of the methods set forth in papers (1,4).

Theorem 3 is proved as follows. Let λ_n be a given sequence tending to zero. Without loss of generality, one may assume that λ_n tends to zero monotonically. Let $\lambda(z) = \lambda_{[z]}$ ($[z]$ is the integer part of z), $\varepsilon'_1 = 1$, and for $n \geqslant 1$

$$\varepsilon'_{n+1} = \frac{8}{n+1}\left[\lambda^{3/2}\left(\frac{n}{16}\right) - \lambda^{3/2}\left(\frac{n+1}{16}\right)\right].$$

Obviously, the series $\sum(n+1)\varepsilon'_{n+1}$ converges.

Put $\sum \varepsilon'_n = E$ and $\varepsilon_n = \varepsilon'_n/E$. Then

$$\sum \varepsilon_n = 1.$$

The distribution function

$$F(x) = \sum_{n=1}^{\infty} \varepsilon_n \Phi\left(\frac{x}{\sqrt{n}}\right),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\,dx,$$

will be the function required by Theorem 3.

This example was constructed for another purpose by Yu. V. Prokhorov and is printed here for the first time with the kind permission of Yu. V. Prokhorov.

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CITED LITERATURE

1. V. Prokhorov, *DAN*, 83, No. 6 (1952).

2. M. Mamatov, *Tr. Inst. matem. AN UzSSR*, vol. 32, 91 (1961).

3. W. L. Smith, *Proc. Cambridge Phil. Soc.*, 49, 462 (1953).

4. S. Kh. Sirazhdinov, *Teoriya veroyatn. i ee primenen.*, 4, 229 (1959).

Note: Figure translations are in progress. See original paper for figures.