MATHEMATICS

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. A. TUPCHIEV

ON THE EXISTENCE, UNIQUENESS, AND ASYMPTOTICS OF THE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

(Presented by Academician I. G. Petrovskii on 20 IX 1961)

Consider the system

$$\mu^2 \frac{d^2 z}{dt^2} = F(z, y, t),$$

$$\frac{dy}{dt} = f(z, y, t), \tag{1}$$

$$z(0) = z(1) = y(0) = 0,$$

where μ is a small parameter.

I. The question of the existence and uniqueness of the solution of system (1) was examined in paper ([1]), but under conditions on the right-hand sides of system (1) in the unbounded domain R:

$$|z| < \infty, \qquad |y| < \infty, \qquad 0 \le t \le 1.$$

Here we eliminate the need to impose conditions on the right-hand sides in all of R and, for this purpose, single out by means of the methods of works ([2,3]) a bounded domain $G \in R$. The result is Theorem 1.

Theorem 1. Suppose that conditions (A) are satisfied:

1. The system obtained from (1) if one formally puts $\mu = 0$ (or, as we shall say, the degenerate system)

$$0 = F(z, y, t), \tag{2'}$$

$$\frac{dy}{dt} = f(z, y, t), \qquad y(0) = 0, \tag{2''}$$

has everywhere on the interval $0 \leq t \leq 1$ a unique solution $\{\bar{z}(t), \bar{y}(t)\}$, corresponding to some isolated root $z_i = \varphi_i(y, t)$ of equation $(2')$, and the derivative $d^2\bar{z}/dt^2$ is bounded on the interval $0 \leq t \leq 1$.

2. The functions $F(z, y, t)$, $f(z, y, t)$ have continuous first partial derivatives with respect to z, y, and are continuous in t in the domain G: $0 \leq t \leq 1$, $|y - \bar{y}(t)| < \varepsilon$, $|z - \bar{z}(t)| < d$, where $d = \max\{|\bar{z}(0)|, |\bar{z}(1)|\} + \varepsilon$; $\varepsilon > 0$ is any arbitrarily small number.

3. In the domain G, $F_z \geq a \geq 0$.

Then, for sufficiently small $\mu \leq \mu_0$, there exists a unique solution of system (1), and the estimates

$$|z - \bar{z}| < |\bar{z}(0)| e^{-\sqrt{m}\, t/\mu} + |\bar{z}(1)| e^{-\sqrt{m}\,(1-t)/\mu} + C\mu,$$

$$|y - \bar{y}| < \frac{|\bar{z}(0)|}{\sqrt{m}}\, \mu + \frac{|\bar{z}(1)|}{\sqrt{m}}\, \mu,$$

hold, where C is a constant independent of μ, $0 < m < a$.

II. The method of constructing the asymptotics is a further development of the method used in paper ([4]). The asymptotics is constructed according to the following scheme.

Let us define three types of functions.

$$\text{a) } \bar{y}_{\mu,n},\ \bar{z}_{\mu,n}; \qquad \text{b) } \overset{(n)}{\bar{y}}_{\mu k},\ \overset{(n)}{\bar{z}}_{\mu k}; \qquad \text{c) } y_n,\ z_n.$$

1. First set $n = 0$.

The functions a) are determined from system (2).

The functions b) are determined in the following way. We write system (1) in the variables μ and $\tau = \dfrac{t - t^0}{\mu}$ (in what follows, t^0 will be assigned the values 0 or 1):

$$\frac{d^2 z}{d\tau^2} = F(z, y, t^0 + \mu\tau), \qquad \frac{dy}{d\tau} = f(z, y, t^0 + \mu\tau). \tag{3}$$

Represent z and y in the form of series

$$z = z_0(\tau) + \bar{z}_1(\tau) + \dots + \mu^n z_n(\tau) + \cdots, \qquad y = y_0(\tau) + \mu y_1(\tau) + \dots + \mu^n y_n(\tau) + \cdots, \tag{3'}$$

whose coefficients are determined from the system of equations obtained as a result of substituting the series ($3'$) into system (3), followed by equating terms with equal powers of μ.

Then z_0, y_0 satisfy the system

$$\frac{d^2 z_0}{d\tau^2} = F(z_0, y_0, t^0), \qquad \frac{dy_0}{d\tau} = 0 \tag{4}$$

or, in the variables t and μ,

$$\mu^2 \frac{d^2}{dt^2} \overset{(0)}{z} = F(\overset{(0)}{z}, \overset{(0)}{y}, t^0), \qquad \frac{d}{dt} \overset{(0)}{y} = 0.$$

Here we introduce the notation $\overset{(n)}{z} = \mu^n z_n$, $\overset{(n)}{y} = \mu^n y_n$. The last expression leads to the system

$$0 = F(\overset{(0)}{\bar{z}}, \overset{(0)}{\bar{y}}, t^0), \qquad \frac{d \overset{(0)}{\bar{y}}}{dt} = 0,$$

from which, under the additional condition $\overset{(0)}{\bar{y}}\Big|_{t=t_0} = \bar{y}(t^0)$, the functions b) are determined. For $k > 0$ we regard the functions b) as identically equal to zero.

We define the functions c). These functions have not yet been defined, since we have not assigned additional conditions to system (4). Instead of defining them directly, we shall determine the difference $\xi_0 = z_0 - \bar{z}_0$ and y_0 from the system

$$\frac{d^2}{d\tau^2}\xi_0 = F(\xi_0 + \bar{z}_0, y_0, t^0), \qquad \frac{dy_0}{d\tau} = 0.$$

We take the additional condition in the form

$$\text{for } t^0 = 0 \qquad \xi_0^0\big|_{\tau=0} = -\overset{(0)}{\bar{z}}{}^0, \qquad \xi_0^0\big|_{\tau=\infty} = 0, \qquad y_0^0\big|_{\tau=0} = 0;$$

$$\text{for } t^0 = 1 \qquad \xi_0^1\big|_{\tau=0} = -\overset{(0)}{\bar{z}}{}^1, \qquad \xi_0^1\big|_{\tau=-\infty} = 0, \qquad y_0^1\big|_{\tau=0} = \bar{y}(1).$$

Here and in what follows, the right superscript will indicate at which value of t^0 ($t^0 = 0$ or $t^0 = 1$) the functions of types b) and c) are taken.

2. Suppose that we have determined functions of types a), b), c) up to the $(n-1)$-st number; we shall now determine them for the n-th number.

We determine the functions a). To this end, differentiating system (1) n times with respect to μ and putting it equal to zero, we obtain

$$n(n-1)\frac{d^2}{dt^2}\overline{z}_{\mu,n-2} = \overline{F}_{\mu n}, \qquad \frac{d}{dt}\overline{y}_{\mu,n} = \overline{f}_{\mu n}.$$

We take the additional condition in the form

$$\overline{y}_{\mu,n}\Big|_{t=0} = (-1)^n \int_0^\infty \tau^n f^0_{n-1,\tau}\, d\tau.$$

We determine the functions b). As a result of substituting the series (3′) into system (3) and equating the terms with μ to the power n, we obtain the system

$$\frac{d^2 z_n}{d\tau^2} = F_n, \qquad \frac{d}{d\tau}y_n = f_{n-1}, \tag{5}$$

or, in the notation $z^{(n)}, y^{(n)}$ and the variables t and μ:

$$\mu^2\frac{d^2}{dt^2}z^{(n)} = F^{(n)}, \qquad \frac{d}{dt}y^{(n)} = f^{(n-1)}. \tag{6}$$

The functions $F^{(n)}$ and $f^{(n-1)}$ can be represented in the form of symbolic formulas

$$F^{(n)} = \frac{1}{n!}\frac{d^n}{dv^n}\exp D\Big|_{v=0} F^{(0)},$$

where

$$D = d_1 v + d_2 v^2 + \ldots + d^n_{nv}, \quad d_1 = z^{(1)}\frac{\partial}{\partial z_0} + y^{(1)}\frac{\partial}{\partial y_0} + (t - t^0)\frac{\partial}{\partial t_0}, \quad d_2 =$$

$$= z^{(2)}\frac{\partial}{\partial z_0} + y^{(2)}\frac{\partial}{\partial y_0}, \ldots, \quad d_n = z^{(n)}\frac{\partial}{\partial z_0} + y^{(n)}\frac{\partial}{\partial y_0}.$$

Similarly for $f^{(n-1)}$.

Differentiating system (6) k times with respect to μ and putting it equal to zero, we arrive at the system

$$k(k-1)\frac{d^2}{dt^2}\overline{z}^{(n)}_{\mu,k-2} = \overline{F}^{(n)}_{\mu k}, \qquad \frac{d}{dt}\overline{y}^{(n)}_{\mu k} = \overline{f}^{(n-1)}_{\mu k},$$

from which, under the additional condition

$$\overline{y}^{(n)}_{\mu k}(t^0) = \begin{cases} \overline{y}_{\mu k}(t^0), & n = k, \\ 0, & n \neq k, \end{cases}$$

the functions b) are determined.

We introduce the following notation:

$$\overline{z}_{nk} = \frac{1}{\mu^{n-k}}\overline{z}^{(n)}_{\mu k}, \qquad \overline{z}_n = \overline{z}_{n0} + \overline{z}_{n1} + \ldots + \frac{1}{n!}\overline{z}_{nn},$$

and, as can be shown, the identity holds

$$\overline{z}_{nk} = \frac{\tau^{n-k}}{(n-k)!}\,\overline{z}_{\mu k, t^{n-k}}(t^0).$$

Similarly for y, and also for f and F.

Now we determine the functions c), as above, in the case $n = 0$, through the difference $\xi_n = z_n - \overline{z}_n$ and y_n. The functions ξ_n and y_n satisfy the system

$$\frac{d^2}{d\tau^2}\xi_n = F_{z_0}\xi_n + G_n, \qquad \frac{d}{d\tau}y_n = f_{n-1},$$

where G_n depends on the functions of preceding indices and on τ.

We take the additional conditions in the form:

$$\text{for } t^0 = 0 \qquad \xi^0_n\big|_{\tau=0} = -\frac{1}{n!}z^{0\,\overline{(n)}}_{\mu n}, \qquad \xi^0_n\big|_{\tau=\infty} = 0, \qquad y^{(0)}_n\big|_{\tau=0} = 0;$$

$$\text{for } t^0 = 1 \qquad \xi^1_n\big|_{\tau=0} = -\frac{1}{n!}z^{1\,\overline{(n)}}_{\mu n}, \qquad \xi^1_n\big|_{\tau=-\infty} = 0,$$

$$y^1_n\big|_{\tau=0} = \frac{1}{n!}\left(\overline{y}_{\mu n}(1) - (-1)^n\int_0^{-\infty}\tau^n f^1_{n-1}\tau^n\,d\tau\right).$$

The existence of the functions $\xi_0, \xi_1, \ldots, \xi_n$ as solutions of the corresponding second-order equations on the semi-infinite interval follows from work ([5]).

Theorem 2. *Suppose that conditions (A) are satisfied and, in addition, the functions $F(z, y, t)$, $f(z, y, t)$ in the domain G possess continuous partial derivatives with respect to all arguments up to orders $(n+1)$ and n, respectively, inclusive; moreover, on the sets $t = 0,\ y = 0,\ |z| < |\overline{z}(0)|$ and $t = 1,\ y = \overline{y}(1),\ |z| < |\overline{z}(1)|$, they respectively possess derivatives with respect to z up to orders $(2n+1)$ and $2n$, inclusive.*

Then the estimates

$$|z - Z_n| < C\mu^{n+1}, \qquad |y - Y_n| < C\mu^{n+1},$$

hold, where C does not depend on t, μ, for sufficiently small $\mu \leqslant \mu_0$ and t varying on the interval $0 \leqslant t \leqslant 1$, while Z_n has the form

$$\begin{aligned} Z_n = {} & z^{0\,\overline{(0)}} + z^{0\,\overline{(1)}} + \dots + z^{0\,\overline{(n)}} + z^{1\,\overline{(0)}} + z^{1\,\overline{(1)}} + \dots + z^{1\,\overline{(n)}} + \overline{z} + \mu\overline{z}_\mu + \dots + \frac{\mu^n}{n!}\overline{z}_{\mu n} \\ & - \left[\left(z^{0\,\overline{(0)}} + z^{0\,\overline{(1)}} + \dots + z^{0\,\overline{(n)}}\right) + \mu\left(z_\mu^{0\,\overline{(1)}} + \dots + z_\mu^{0\,\overline{(n)}}\right) + \dots + \frac{\mu^n}{n!} z_{\mu n}^{0\,\overline{(n)}}\right] \\ & - \left[\left(z^{1\,\overline{(0)}} + z^{1\,\overline{(1)}} + \dots + z^{1\,\overline{(n)}}\right) + \mu\left(z_\mu^{1\,\overline{(1)}} + \dots + z_\mu^{1\,\overline{(n)}}\right) + \dots + \frac{\mu^n}{n!} z_{\mu n}^{1\,\overline{(n)}}\right]. \end{aligned}$$

The expression for Y_n is analogous.

Moscow State University
named after M. V. Lomonosov

Received
13 IX 1961

REFERENCES

[1] T. G. Bektashi, *Uch. zap. Azerb. univ.*, No. 5, 3 (1958).
[2] M. Nagumo, *Proc. Phys. Math. Soc. Japan*, Ser. III, **19**, 861 (1937).
[3] N. I. Brish, *DAN*, **95**, No. 3 (1954).
[4] A. B. Vasil' eva, V. A. Tupchiev, *DAN*, **135**, No. 5 (1960).
[5] Yu. A. Klykov, Dissertation, *The limiting boundary-value problem for an ordinary second-order differential equation*, Moscow State University, 1959.

Note: Figure translations are in progress. See original paper for figures.