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Abstract

Full Text

MATHEMATICS

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ON EQUATIONS OF MIXED TYPE IN THREE-DIMENSIONAL DOMAINS

In the present note a three-dimensional analogue of the well-known Tricomi problem is proposed and studied.

Denote by D a finite simply connected three-dimensional domain bounded by the piecewise smooth surface $S^* : z = f(x, y) \geqslant 0$ and by two cones $S_1 : x + x_0 = \sqrt{y^2 + (z - z_0)^2}$ and $S_2 : x - x_0 = -\sqrt{y^2 + (z - z_0)^2}$, $x_0 > z_0 \geqslant 0,\ z \leqslant 0$.

Let $D_1 = (D \cap z < 0)$, $D_2 = (D \cap z > 0)$, and let T be the domain of the plane $z = 0$ separating D_1 from D_2.

The three-dimensional analogue of the Tricomi problem is the following

Basic mixed problem. *Find a function $U(x, y, z)$, continuous in the closed domain $\bar{D}$, with continuous first derivatives inside D, satisfying the equation*

$$LU = \operatorname{sgn} z \cdot U_{xx} + U_{yy} + U_{zz} = g \qquad (1)$$

in the domain D for $z \neq 0$, and vanishing on S^* and on one of the characteristic cones S_1 or S_2, for example on S_1. It is assumed that $g(x, y, z)$ is a given sufficiently smooth function.

Finding a regular solution $U(x, y, z)$ in the domain D_2 of equation (1), satisfying the conditions $U = 0$ on S^*, $U_z = 0$ (or $U = 0$) on T, is a classical problem of potential theory, and it always has, moreover, a unique solution.

The problem of determining a regular solution $U(x, y, z)$ in the domain D_1 of equation (1), satisfying the conditions $U = 0$ on S_1 (or on S_2) and $U_z = 0$ (or $U = 0$) on T, is a special case of the problem considered by S. L. Sobolev ([1]).

In the case $x_0 = 1,\ z_0 = 0$, under the requirement of square summability of all first derivatives of the sought solution both over the volume and over the surface of the domain D_1, the uniqueness of the solution of this problem immediately follows from the easily verified identity

$$2 \iiint_{D_1} (x-1) g U_x \, dv = \iiint_{D_1} (U_x^2 + U_y^2 + U_z^2) \, dv + \frac{1}{\sqrt{2}} \iint_{S_2} \left[(1 - x)(U_x^2 + U_y^2 + U_z^2) - 2yU_xU_y - 2zU_xU_z \right] ds.$$

By a **weak solution** of the indicated problem we shall mean a function $U(x, y, z)$ with derivative U_z from the Hilbert space H, satisfying the identity

$$\iiint_{D_1} W_z g\, dv = -\iiint_{D_1} U_z LW\, dv, \tag{2}$$

where $W(x, y, z)$ is an arbitrary function of class C^2 satisfying the conditions: $W_x = 0$ on S_1, $W = 0$ on T and on S_2.

The **existence** of a function $U(x, y, z)$ satisfying identity (2) follows from the presence of the inequality

$$\iiint_{D_1} W_z^2\, dv \leqslant k \iiint_{D_1} (LW)^2\, dv, \tag{3}$$

where k is a positive constant independent of W.

The validity of inequality (3), in turn, is a consequence of the identity

$$2 \iiint_{D_1} (1+x)\, W_x LW\, dv =$$

$$= \iiint_{D_1} (W_x^2 + W_y^2 + W_z^2)\, dv + \frac{1}{\sqrt{2}} \iint_{S_1} (1+x)(W_y^2 + W_z^2)\, ds.$$

In the study of the principal mixed problem we shall assume that S^* consists of two conical surfaces $S_3 :\ x - x_0 = -\sqrt{y^2 + (z+z_0)^2}$ and $S_4 :\ x + x_0 = \sqrt{y^2 + (z+z_0)^2},\ z \geqslant 0$.

For the solution $U(x, y, z)$ of this problem, in the case $x_0 = 1,\ z_0 = 0$, the following integral equalities hold (provided, of course, that the corresponding improper integrals exist):

$$2 \iiint_{D_2} [(x-1)U_x + yU_y + zU_z]\, g\, dv = \iiint_{D_2} (U_x^2 + U_y^2 + U_z^2)\, dv +$$

$$+ \sqrt{2} \iint_{S_4} (U_x^2 + U_y^2 + U_z^2)\, ds + 2 \iint_T [(1-x)U_{xU} z - yU_{yU} z]\, dx\, dy, \tag{4}$$

$$2 \iiint_{D_1} [(x-1)U_x + yU_y]\, g\, dv =$$

$$= 2 \iiint_{D_1} U_z^2\, dv - 2 \iint_T [(1-x)U_{xU} z - yU_{yU} z]\, dx\, dy +$$

$$+\frac{1}{\sqrt{2}}\iint_{S_2}\frac{1}{1-x}\left\{[(1-x)U_y-yU_x]^2+[(1-x)U_x-yU_y-zU_z]^2\right\}ds. \qquad (5)$$

From equalities (4) and (5), taking into account that the first derivatives of the desired solution must be continuous in the domain D, there follows the uniqueness of the solution of the principal mixed problem.

We note that the method proposed here for proving uniqueness of the solution of the principal mixed problem is quite artificial. Nevertheless, there exists a rather broad class of surfaces S^* for which, by this method, one can prove uniqueness of the solution of the indicated problem. For this purpose, in the integrands of formulas (4) and (5), the factors $x-1$, y, and z should be replaced by suitably chosen functions.

A function $U(x,y,z)$ with derivative U_z from the Hilbert space H will be called a **weak solution** of the principal mixed problem if the identity

$$\iiint_D W_z g\,dv=-\iiint_D U_z\,(\operatorname{sgn} z\cdot W_{xx}+W_{yy}+W_{zz})\,dv \qquad (6)$$

holds for every function W belonging in each of the domains D_1 and D_2 to the space C^3 and satisfying the conditions: $W_y=W_z=0$ on S_3; $W_x=0$ on S_1; $W_x=W_z=0$ on S_4; $W=W_z=0$ on S_2; $(W_{yy}+W_{zz})_{z=-0}=(W_{yy}+W_{zz})_{z=+0}$, $(W_x)_{z=-0}=-(W_x)_{z=+0}=0$, $(W_y)_{z=-0}=(W_y)_{z=+0}$, $(W_z)_{z=-0}=(W)_{z=+0}$, and $LW=0$ on the boundary of the domain D.

The existence of a function $U(x,y,z)$ for which identity (6) holds is a consequence of the inequality

$$\iiint_D W_z^2\,dv\leqslant k\iiint_D (LW)^2\,dv,$$

whose validity in the case $x_0=1$, $z_0=0$ is again easily concluded from the evident inequality

$$\iiint_{D_2}[m(1+z)W_x+yW_y+zW_z]\,LW\,dv+\iiint_{D_1}[m(1+x)W_x+yW_y]\,LW\,dv\geqslant$$

$$\geq\iiint_{D_2}\left[(2-m)W_x^2+mW_y^2+mW_z^2\right]\,dv+$$

$$+\iiint_{D_1}\left[(m-1)W_x^2+(m-1)W_y^2+(m+1)W_z^2\right]\,dv,$$

where m is a constant, with $1 < m < 2$.

With regard to extending the method presented here for proving the existence of weak solutions to the case of a certain class of surfaces S^*, one may repeat the argument given above in connection with the proof of uniqueness of the solution.

It follows from identity (6) that a sufficiently smooth weak solution of the basic mixed problem is an ordinary solution.

Under the requirement of continuous differentiability of the right-hand side $g(x, y, z)$ of equation (1), the weak solution obtained above indeed has the required order of smoothness, but this fact is far from obvious and requires a separate proof.

Let us now consider the equation

$$zU_{xx} + U_{yy} + U_{zz} = g, \tag{7}$$

where $g(x, y, z)$ is a given smooth function, and the domain D has boundary $S = S_1 + S_2 + S_3$, where $S_1 : \frac{2}{3}(-z)^{3/2} - x = 1;\quad S_2 : \frac{2}{3}(-z)^{3/2} + x = 1,\ z \le 0$, and $S_3 : x^2 + \frac{4}{9}z^3 = 1,\ z \ge 0$.

The following problem may also be regarded as a certain analogue of the Tricomi problem:

Find a function $U(x, y, z)$ that is continuous in the closed domain $\overline{D}$, satisfies equation (7) inside D, and vanishes on S_1 and S_3.

The method applied above makes it possible to establish existence and uniqueness of the solution only to the problem just formulated under the additional requirement that the function $g(x, y, z)$ and the sought solution $U(x, y, z)$ decay (to a specified order) as $y \to \pm\infty$. In studying this problem, the method of integral transformations can also be applied successfully; moreover, as S_3 one may take an arbitrary smooth cylindrical surface $z = f(x)$ resting on the straight lines $z = 0,\ x = -1$ and $z = 0,\ x = 1$.

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CITED LITERATURE

1. S. L. Sobolev, *Matem. sborn.*, **11** (53), 3, 155 (1942).

Note: Figure translations are in progress. See original paper for figures.