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Abstract

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ON BOUNDARY VALUE PROBLEMS FOR PARABOLIC SYSTEMS IN A HALF-SPACE

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The present note is devoted to the study of the fundamental matrix of solutions (the Green matrix) of a general boundary value problem for a parabolic system in a half-space in the simplest case: the coefficients of the system and of the boundary condition are assumed to be constant, and the boundary conditions and the system contain operators homogeneous (in the parabolic sense). This study makes it possible, using well-developed methods, to investigate boundary value problems for systems of general form (variable coefficients and the presence, in the system and in the boundary conditions, of derivatives of all orders) for domains with smooth boundaries. We continue the investigations of T. Ya. Zagorskii (^{1}), refining and generalizing the results obtained therein in the following directions: 1) boundary conditions of arbitrary order are included; 2) sharp estimates of the Green matrix of the boundary value problem are obtained; 3) the possibility of extending the Green matrix into the half-space $x_n < 0$ is investigated, which plays an essential role in the study of boundary value problems for domains of general form; 4) for problems of the type of the first boundary value problem, a representation of the Green matrix is obtained in terms of derivatives of the fundamental matrix of solutions of the system; 5) sharp theorems on the solvability of the problem in the half-space are established.

The exposition is carried out for systems parabolic in the sense of I. G. Petrovskii; however, the results are also valid for the $\overrightarrow{2b}$-parabolic systems defined by the author in (^{2}), since the constructions and proofs presented are in fact based on the hypoellipticity of the boundary value problems under consideration, studied in the case of one equation by L. Hörmander (^{3}), and on the generalized homogeneity of the systems and boundary conditions under consideration.

1. Consider the following boundary value problem:

$$\frac{\partial u}{\partial t} = \sum_{|k|=2b} A_k(-iD_x)^k u \equiv A(iD_x)u; \tag{1}$$

$$u\big|_{t=+0} = 0, \qquad x_n > 0, \quad -\infty < x_s < \infty, \quad s = 1, 2, \dots, n-1,$$
$$x' = (x_1, x_2, \dots, x_{n-1}); \tag{2}$$

$$B_j\left(\frac{\partial}{\partial t},\, iD_x\right) u\Big|_{x_n=+0} = \sum_{m=1}^{N}\sum_{2bl_0+|l|=r_j} B_{jm}^{(l_0 l)}\frac{\partial^{l_0}}{\partial t^{l_0}}(-iD_x)^l u_m\Big|_{x_n=+0} = f_j(x',t),$$

$$j = 1, 2, \dots, bN; \tag{3}$$

N is the number of equations in system (1), and the coefficients A_k are assumed real. Problem (1)–(3) is studied under the following basic assumption:

$$\det\left\{\int_{\Gamma^+} B(\sigma,\rho)\,(A(\sigma)-\rho E)^{-1}\,(E,\sigma_n E,\dots,\sigma_n^{b-1}E)\;d\sigma_n\right\} \neq 0 \tag{4}$$

for arbitrary real $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and $\rho = -\delta_1|\sigma'|^{2b} + a + ip_1$; a is any positive number; $|\sigma'|^2 = \sigma_1^2 + \dots + \sigma_{n-1}^2$; $-\infty < p_1 < \infty$; δ_1 is some positive constant; Γ^+ is a contour in the complex σ_n-plane enclosing all σ_n-roots of the equation $\det\{A(\sigma)-\rho E\} = 0$.

with $\operatorname{Im}\sigma_n > 0$,

$$B(\sigma,p) = \left\|\sum_{2bl_0+|l|=r_j} B_{jm}^{(l_0 l)} p^{l_0}\sigma^l\right\|_{\substack{j=1,2,\dots,bN\\ m=1,2,\dots,N}};$$

$$(E,\sigma_{nE},\dots,\sigma_n^{b-1}E) = \begin{pmatrix} 1 & 0 & \sigma_n & 0 & \cdots & \sigma_n^{b-1} & 0\\ 0 & 1 & 0 & \sigma_n & \cdots & 0 & \sigma_n^{b-1}\end{pmatrix}.$$

Assuming at first infinite differentiability and finiteness of the functions $f_i(x',t)$, applying the Fourier transform in x' and the Laplace transform in t, and then changing the order of integration, we arrive at the following formula, which gives the solution of problem (1)–(3):

$$u_i(x,t) = \int_0^t d\tau \int \sum_{m=1}^{bN} G_{im}(t-\tau, x-\xi') f_m(\xi',\tau)\,d\xi', \tag{5}$$

where

$$G(t,x) = \frac{1}{(2\pi)^n i}\int e^{i(x',\sigma')}\,d\sigma' \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,dp \int_{\Gamma^+} e^{ix_n\sigma_n}(A(\sigma)-pE)^{-1}(E,\sigma_{nE},\dots$$

$$\dots,\sigma_n^{b-1}E)\,d\sigma_n\left[\int_{\Gamma^+} B(\sigma,p)(A(\sigma)-pE)^{-1}(E,\sigma_{nE},\dots,\sigma_n^{b-1}E)\,d\sigma_n\right]^{-1};$$

$$\gamma = -\delta_1 |\sigma'|^{2b} + a_1; \tag{6}$$

a_1 is some fixed positive constant.

2. The following theorem describes the properties of the Green matrix.

Theorem 1. 1) If the solvability condition (4) is fulfilled, then for $t > 0$ the matrix $G(t,x)$ can be analytically continued to the whole space $x_1, x_2, \dots, x_n$. In this case $G(t,x)$ will be an analytic function of $x_1, x_2, \dots, x_n$, infinitely differentiable with respect to all its arguments; for the derivatives of $G(t,x)$ the estimates

$$\left|\frac{\partial^{m_0}}{\partial t^{m_0}} D_x^m G_{ij}(t,x)\right| \le$$

$$\le C_{m_0 m} t^{-\frac{n+2b-1-r_j+2bm_0+|m|}{2b}} \exp\left\{-c_1 \left|\frac{x'}{t^{1/2b}}\right|^q - c_2 \operatorname{sign} x_n \left(\frac{x_n}{t^{1/2b}}\right)^q\right\}; \tag{7}$$

c_1 and c_2 are positive constants; $q = \dfrac{2b}{2b-1}$; $|x'|^2 = x_1^2 + \dots + x_{n-1}^2$.

Moreover, for $x_n \ge 0,\ t > 0$ the following estimate is also valid:

$$\left|B_i\left(\frac{\partial}{\partial t}, iD_x\right) G_{ij}(t,x)\right| \le C t^{-\frac{n+2b+r_i-r_j}{2b}} x_n \exp\left\{-c\left|\frac{x}{t^{1/2b}}\right|^q\right\}. \tag{8}$$

2) If derivatives with respect to x_n of order higher than $b-1$ enter into the boundary conditions, then the columns of the matrix $G(t,x)$ are represented in the form

$$G_j(t,x) = \sum_{|k| \le 2b-1-r_j} D_x^k G_0(t,x) A_{kj}, \qquad 0 \le r_j \le 2b-1; \tag{9}$$

$$\frac{\partial^{m_0}}{\partial t^{m_0}} D_x^m G_j(t,x) = G_0(t,x) A_{0j}^{(m_0 m)}, \qquad r_j \ge 2b, \quad 2bm_0 + |m| = r_j - 2b + l.$$

A_{kj}, $A_{0j}^{(m_0 m)}$ are constant one-column matrices, and $G_0(t,x)$ is the fundamental matrix of solutions of system (1).

Before turning to the description of how Theorem 1 is established, let us note some qualitative consequences of it. It is known that the fundamental matrix of solutions of system (1) is determined by the formula

$$G_0(t,x) = \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int e^{i(x,\sigma)+A(\sigma)t}\, d\sigma, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and is a solution of this system with a point singularity (at the point $(0,0)$). Inequality (7) shows that the Green matrix $G(t,x)$, being, naturally, extended by zero for $t \leqslant 0$, has as possible singular points points lying in the cone $|x'| \leqslant [c_2/c_1]^{1/q}|x_n|$, $x_n < 0$, in the plane $t = 0$. Assertion 2) of the theorem singles out a class of boundary conditions of the type of the first boundary-value problem for which the Green matrix, or some of its derivatives, have a point singularity (in the latter case, derivatives of lower order are regular everywhere). Inequalities (7), (8) are sharp in the sense that, in the case of the first and second boundary-value problems for the heat equation, they become equalities.

The proof of the first part of Theorem 1 is based on lemmas of an algebraic character, which make it possible to describe the domain of analyticity of the matrix

$$Q(\sigma,p) = (A(\sigma) - pE)^{-1}(E, \sigma_{nE}, \dots, \sigma_n^{b-1}E) \times$$

$$\times \left[\int_{\Gamma^+} B(\sigma,p)(A(\sigma) - pE)^{-1}(E, \sigma_{nE}, \dots, \sigma_n^{b-1}E)\, d\sigma_n \right]^{-1},$$

and the generalized homogeneity of the columns of the matrix $Q(\sigma,p)$. In extending the matrix $G(t,x)$, initially defined only for $x_n > 0$, to the entire space $x_1, x_2, \dots, x_n$ for $t > 0$, essential use is made of the possibility of replacing integration along the straight line parallel to the imaginary axis in the variable p by integration along two rays issuing from the point $a_1 + i0$ and forming with the imaginary axis a sufficiently small angle φ_0 for $\operatorname{Im} p > 0$ and $-\varphi_0$ for $\operatorname{Im} p < 0$, $\varphi_0 > 0$. To establish the second part of the theorem, we note that if the boundary conditions include derivatives with respect to x_n of order not higher than $b-1$, then in formula (6) one may everywhere pass from integration over Γ^+ to integration over the real axis in the σ_n-plane.

If one first carries out the above-described change of integration with respect to p, which ensures the continuation of the matrix $G(t,x)$, then the Green matrices for the half-spaces $x_n > 0$ and $x_n < 0$ coincide; therefore, for all $x_1, x_2, \dots, x_n$,

$$\left| \frac{\partial^{m_0}}{\partial t^{m_0}} D_x^m G_{ij}(t,x) \right| \leqslant C_{m_0 m} t^{-\frac{n+2b-1-r_j+2bm_0+|m|}{2b}} \exp\left\{ -c \left(\frac{|x|}{t^{1/2b}} \right)^q \right\}.$$

From this and from the theorem on the behavior of solutions in a neighborhood of an isolated singular point[4], representation (9) follows.

3. Estimates (7), (8) make it possible to establish exact theorems on the solvability of problem (1)–(3) in classes of rapidly increasing functions. We give one of them.

Theorem 2. Let the functions $f_m(x', t)$ satisfy the following conditions: 1) they are defined in the strip $\Pi_T\{0 \leqslant t \leqslant T,\ -\infty < x_s \leqslant \infty,\ s = 1, 2, \dots, n-1\}$, have in it $\bar{r} - r_m$, $\bar{r} = \max_j r_j$, derivatives with respect to $x_1, x_2, \dots, x_{n-1}$, and locally satisfy a Hölder condition in t with exponent $\frac{\bar{r} - r_m}{2b}$*; 2) they satisfy the inequality $|f_m(x', t)| \leqslant C_1 \exp\{b|x'|^q\}$.

Then the functions $u_i(x, t)$, defined by formula (5), give a solution of problem (1)–(3) and satisfy the estimate $|u_i(x, t)| \leqslant C_2 \exp\{b_1|x'|^q\}$.

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CITED LITERATURE

[1] T. Ya. Zagorskii, Ukr. Mat. Zh., **9**, No. 3 (1957); DAN, **106**, No. 1 (1956).
[2] S. D. Eidelman, DAN, **133**, No. 1 (1960).
[3] L. Hörmander, Acta Math., **99**, 225 (1958).
[4] S. D. Eidelman, DAN, **125**, No. 4 (1959).

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* If $\bar{r} - r_m > 2b$, then there are $\left[\frac{\bar{r} - r_m}{2b}\right]$ derivatives with respect to t, satisfying a Hölder condition in t wit

Note: Figure translations are in progress. See original paper for figures.

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