Reports of the Academy of Sciences of the USSR

1962

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1962. Volume 143, No. 5

MATHEMATICS

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ON THE PROBLEM OF EFFECTIVENESS AND INEFFECTIVENESS OF REGULAR MATRICES

(Presented by Academician A. N. Kolmogorov on 8 XII 1961)

Put

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k,$$

where $(s_0, s_1, s_2, ...) = \{s_n\}$ is some sequence; $A = \{a_{nk}\}$, $n, k = 0, 1, 2, ...$, is a real regular matrix, i.e. a matrix satisfying the Toeplitz conditions (see, for example, ([1]), p. 79). If $\lim_{n\to\infty} \sigma_n = s$, then the sequence $\{s_n\}$ is summed by the matrix A to the number s. (We shall say that s is the A-limit of the sequence $\{s_n\}$, $s = A\lim\{s_n\}$.) A linear combination with coefficients $\lambda_1, \lambda_2, \dots, \lambda_n$ of the sequences

$$\{s^1\} = (s_0^1, s_1^1, s_2^1, \dots, s_n^1, ...), \quad \{s^2\} = (s_0^2, s_1^2, \dots, s_n^2, ...), \dots, \quad \{s^n\} = (s_0^n, s_1^n, \dots, s_n^n, ...)$$

is called the sequence

$$\{\lambda_1 s^1 + \lambda_2 s^2 + \dots + \lambda_{ns}^n\} = (\lambda_1 s_0^1 + \lambda_2 s_0^2 + \dots + \lambda_{ns} 0^n,\ \lambda_1 s_1^1 + \lambda_2 s_1^2 + \dots + \lambda_{ns} 1^n, \dots,\ \lambda_1 s_n^1 + \lambda_2 s_n^2 + \dots + \lambda_{ns} n^n, ...).$$

The set of all sequences summable by a regular matrix A is called the effective domain of the matrix (see ([1]), Ch. 7); in particular, all convergent sequences belong to the effective domain, forming its so-called trivial part. The effectiveness of a regular matrix is characterized by the size of the nontrivial part of its effective domain. Naturally, the question arises of how large this part is, i.e. how large is the set of divergent sequences summable by the given regular matrix. Similarly, the size of the set of all sequences not summable by a regular matrix is a measure of the ineffectiveness of this matrix; naturally, the question arises about the size of this set, i.e. how large is the set of divergent sequences not summable by the given regular matrix.

The following theorem clarifies the question of the effectiveness of a regular matrix.

Theorem 1. *For an arbitrary regular matrix $A = \{a_{nk}\}$, $n, k = 0, 1, 2, ...$, summing at least one bounded divergent sequence, there exists a continuum set of bounded sequences, divergent simultaneously with any of their finite nontrivial linear (i.e. with coefficients not simultaneously equal to zero) combinations, such that every sequence belonging to this set is also summed by this matrix.*

Thus, the nontrivial part of the effective domain of a regular matrix summing at least one bounded divergent sequence contains a continuum set of bounded sequences divergent simultaneously with any of their finite nontrivial linear combinations.

It follows from this that the effective domain of a matrix summing at least one bounded divergent sequence is of continuum cardinality, since the set of all possible sequences of real numbers (both summable and nonsummable) is of continuum cardinality.

In connection with the theorem under consideration, we note that among regular matrices that do not sum any bounded divergent sequence there are both matrices whose effective domain contains

only a trivial part (for example, the identity matrix $a_{nk} = 1$ $(n = k)$, $a_{nk} = 0$ $(n \neq k)$), and also matrices the nontrivial part of whose effective domain contains an arbitrary finite number of divergent sequences unbounded simultaneously with any of their linear combinations. An example of such a matrix is given in ([2]).

Corollary 1. Whatever regular matrix A may be that sums at least one bounded divergent sequence, and whatever continuum set E of the real axis may be, there exists a continuum set of real bounded sequences, divergent simultaneously with any of their finite linear combinations, such that the set of numbers that are the A-limits of the sequences of this set coincides with the set E. In particular, any interval of the real axis may be taken as the set E.

Every complex sequence $\{s_n\}$ has the form $\{s_n\} = \{s'_n\} + i\{s''_n\}$, where $\{s'_n\}$ and $\{s''_n\}$ are real sequences; $A \lim\{s_n\} = \alpha + i\beta$ ($\{s_n\}$ a complex sequence) if and only if $A \lim\{s'_n\} = \alpha$ and $A \lim\{s''_n\} = \beta$.

With respect to the complex divergent sequences $\{s_n\}$ considered below, it is assumed that the sequences $\{s'_n\}$ and $\{s''_n\}$ forming their real and imaginary parts are also divergent.

Corollary 2. Whatever regular matrix A may be that sums at least one bounded real (or complex) divergent sequence, and whatever two real continuous functions $\varphi(t)$ and $\psi(t)$, $0 \leq t \leq 1$, may be, there exists a continuum set of bounded complex sequences, divergent simultaneously with any of their nontrivial linear combinations, such that the A-limits of the sequences belonging to this set completely fill the Jordan continuum

$$x + iy = \varphi(t) + i\psi(t), \quad 0 \leq t \leq 1.$$

Remark. If the functions $x = \varphi(t)$ and $y = \psi(t)$ realize a mapping of the unit segment onto the unit square (i.e. the curve $\varphi(t) + i\psi(t)$ is a Peano curve), then the totality of the A-limits of the sequences belonging to the continuum set of sequences under consideration fills the unit square. It is natural to ask what an arbitrary connected plane continuum must be like in order that it can be a "carrier" of the totality of A-limits of the continuum set of sequences under consideration. A sufficient condition is that this plane continuum be a continuous image of a closed segment. The corresponding sufficient conditions for this, due to Sierpiński, Hahn, and Mazurkiewicz (for example, "local connectedness," see ([3]), p. 947), cover a rather broad class of geometric objects.

The proof of Theorem 1 is based, in particular, on the following theorem, which is also of independent interest.

Theorem 2. *Let $A = \{a_{nk}\}$, $n, k = 0, 1, 2, \ldots$, be a regular matrix and let $\{s_n\}$ be a bounded divergent sequence summable by the matrix A to zero. Let $\{u_k\}$ be some arbitrary fixed bounded, nonnegative, divergent sequence containing an infinite subsequence of zeros. Suppose, moreover, that the sequence $\{u_k\}$ satisfies the condition*

$$\lim_{k\to\infty} \{u_{k+1} - u_k\} = 0.$$

Then there exist: 1) an unboundedly increasing sequence of integers $\{m_k\}$ (any arbitrary unboundedly increasing sequence of integers contains such a sequence); 2) a nondecreasing integer-valued function $\lambda(m)$; 3) an increasing integer-valued function $\mu(m)$, with $\mu(m) > \lambda(m)$, $m = 0, 1, 2, \ldots$, $\lambda(m_k) = \mu(m_{k-1})$,*

* Functions similar to $\lambda(m)$ and $\mu(m)$ were also considered by Agnew ([4]) and Petersen ([5]).

$k = 1, 2, 3, \ldots$, such that the sequence $\{s_n^*\}$, defined in the following way: $s_n^* = u_k s_n$, $\lambda(m_k) \le n < \mu(m_k)$, $k = 1, 2, 3, \ldots$, will be bounded, divergent, and will be summed by the matrix A to zero.

As for the question, formulated at the beginning of the article, concerning the size of the set of sequences not summable by a given regular matrix, in this direction G. Steinhaus had already shown (see ([6])) that, whatever the regular matrix, there exists at least one bounded divergent sequence not summable by this matrix. (Thus the question of the existence of a universal linear method of summability was answered in the negative.) From the results of A. Brudno ([7]) it follows that, whatever the integer n, there are always n bounded sequences diverging simultaneously with every nontrivial linear combination of them.

The author ([2]) showed that for any regular matrix there exists a countable set of bounded sequences, diverging simultaneously with every linear combination of them, such that no sequence belonging to this set is summable by the given regular matrix. It turns out that this set is continuum in cardinality; namely, the following result, dual in a known sense to Theorem 1, holds.

Theorem 3. *For any regular matrix there exists a continuum set of bounded* *sequences such that every finite nontrivial linear combination of sequences belonging to this set is not summable by this matrix.*

Received
28 IX 1961

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Note: Figure translations are in progress. See original paper for figures.