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Abstract

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Conditions for B-Completeness of Ultrabarrelled and Barrelled Spaces

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The note establishes necessary and sufficient conditions for B-completeness of ultrabarrelled and barrelled spaces. All spaces considered are assumed to be Hausdorff.

Definition 1 ([1]). A topological linear space X is called B-complete if every continuous and nearly open linear mapping* of it onto any topological linear space Y is open**.

Theorem 1. The following assertions about a topological linear space X are equivalent:

1) X is B-complete;
2) every nearly open linear mapping with closed graph of the space X onto any topological linear space Y is open;
3) every nearly continuous linear mapping*** with closed graph of any topological linear space Y into the quotient space of the space X by a closed subspace is continuous.

This theorem is essentially contained in ([1]); a proof of the analogous theorem for locally convex spaces is given in ([6]).

Definition 2 ([7]). A topological linear space (X, t) is called ultrabarrelled if the topology t majorizes any linear topology on X that possesses a fundamental system of t-closed neighborhoods of zero.

Theorem 2. The following assertions about an ultrabarrelled space X are equivalent:

1) X is B-complete;
2) every continuous linear mapping of the space X onto any ultrabarrelled space Y is open;
3) every linear mapping with closed graph of the space X onto any ultrabarrelled space Y is open;
4) every linear mapping with closed graph of any ultrabarrelled space Y into the quotient space of the space X by a closed subspace is continuous;

5) every linear mapping with closed graph of any ultrabarrelled space Y onto the quotient space of the space X by a closed subspace is continuous.

Proof. The implication 1) $\Rightarrow$ 4) follows from Theorem 1 and from the fact that a linear mapping of an ultrabarrelled space is nearly continuous. The implications 4) $\Rightarrow$ 5) and 3) $\Rightarrow$ 2) are obvious.

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* A mapping is called nearly open if the closure of the image of every neighborhood is a neighborhood.

** Locally convex B-complete spaces were introduced by Pták ([2]) (see also ([3–6])).

*** A mapping is called nearly continuous if the closure of the preimage of every neighborhood is a neighborhood.

Suppose that the space X satisfies condition 5), and let us show that then condition 3) also holds for it. Let φ be a linear mapping with closed graph from the space X onto an ultrabarrelled space Y. Then $\varphi^{-1}(0)$ is a closed subspace in X; denote it by H. The mapping φ is the composition of two mappings $\varphi = \psi \circ h$: the canonical homomorphism h of the space X onto the quotient space X/H and a one-to-one linear mapping ψ of the space X/H onto the space Y, while the graph of the mapping ψ is closed in $X/H \times Y$. Consider the mapping ψ^{-1} of the space Y onto the space X/H; its graph is closed in $Y \times X/H$. By condition 5), ψ^{-1} is continuous, and hence ψ is open. Then φ is open as the composition of two open mappings. Thus, the implication 5) $\Rightarrow$ 3) is established.

To complete the proof it remains to show that condition 1) follows from condition 2). Let the space X satisfy condition 2), and let φ be an almost open continuous linear mapping of the space X onto a topological linear space Y. We shall show that Y is ultrabarrelled. Denote by t_Y (t_X) the topology of the space Y (X). Let t be an arbitrary linear topology on Y possessing a fundamental system of t_Y-closed neighborhoods of zero. Consider on X the topology $\varphi^{-1}(t)$, for which the neighborhoods of zero are sets of the form $\varphi^{-1}(V)$, where V is a neighborhood of zero of the space (Y, t). The sets $\varphi^{-1}(V)$, where V runs through a fundamental system of t_Y-closed neighborhoods of zero of the space (Y, t), form a fundamental system of t_X-closed neighborhoods of zero for the linear topology $\varphi^{-1}(t)$ on X. From the ultrabarrelledness of X it then follows that $\varphi^{-1}(V)$ is a neighborhood of zero in (X, t_X); denote it by U. Since φ is almost open, the closure of the set $\varphi(U)$ in Y is a neighborhood of zero, but

$$\overline{\varphi(U)} = \overline{\varphi[\varphi^{-1}(V)]} = \overline{V} = V.$$

Then V is a neighborhood of zero in (Y, t_Y), i.e. t_Y majorizes t, and this means that Y is ultrabarrelled. From condition 2) it then follows that φ is open. The theorem is proved.

Remark. Theorem 2 remains valid if everywhere in its formulation the ultrabarrelled space is replaced by a locally convex ultrabarrelled space.

Theorem 3. The following assertions about a barrelled space X are equivalent:

1) X is B-complete;

2) every continuous linear mapping of the space X onto any barrelled space Y is open;

3) every linear mapping with closed graph of the space X onto any barrelled space Y is open;

4) every linear mapping with closed graph of any barrelled space Y into the quotient space of the space X by a closed subspace is continuous;

5) every linear mapping with closed graph of any barrelled space Y onto the quotient space of the space X by a closed subspace is continuous.

The proof is analogous to the proof of Theorem 2.

Definition 3. A topological linear space X is called B_r-complete if every one-to-one continuous and almost open linear mapping of it onto any topological linear space Y is open*.

* Locally convex B_r-complete spaces were introduced by Pták (2) (see also (3)).

Theorem 4. The following assertions about the ultrabarrelled space X are equivalent:

1) X is B_r-complete;

2) every one-to-one continuous linear mapping of the space X onto any ultrabarrelled space Y is open;

3) every one-to-one linear mapping with closed graph of the space X onto any ultrabarrelled space Y is open;

4) every linear mapping with closed graph of any ultrabarrelled space Y into the space X is continuous;

5) every linear mapping with closed graph of any ultrabarrelled space Y onto the space X is continuous.

Analogous theorems are valid for barrelled and locally convex ultrabarrelled spaces.

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Note: Figure translations are in progress. See original paper for figures.