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Abstract

Full Text

MATHEMATICS

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RADICALS ASSOCIATED WITH REPRESENTATIONS OF GROUPS

(Presented by Academician A. I. Mal' tsev, 23 XII 1961)

Let a certain representation f of a group Γ by automorphisms of a group \mathfrak{G} be given. We define the radical of this representation in the group Γ . A pair of subgroups H and F ($H \subset F$) of \mathfrak{G} will be called a Γ -**composition pair** if in the group \mathfrak{G} there is a Γ -composition normal system in which the subgroups H and F constitute a jump. In this case the factor group F/H will be called a Γ -**composition factor**. It is not hard to see that, if the representation of Γ contains all inner automorphisms of the group \mathfrak{G} , then the pair H, F will be a Γ -composition pair if and only if H and F are Γ -admissible normal divisors of the group \mathfrak{G} and the factor group F/H contains no proper Γ -admissible normal divisors of the group \mathfrak{G}/H . In every Γ -composition factor F/H the representation f induces a certain irreducible representation of the group Γ . Denote by $\mathfrak{Z}(F/H)$ the kernel of such a representation, and by $a(\Gamma, \mathfrak{G})$ we shall denote the intersection of all $\mathfrak{Z}_\Gamma(F/H)$ over all Γ -composition pairs of the group \mathfrak{G} . The subgroup $a(\Gamma, \mathfrak{G})$ will be called the radical of the representation. It is clear that the factor group $\Gamma/a(\Gamma, \mathfrak{G})$ is a subdirect product of (irreducible) groups of automorphisms induced by the group Γ in the various Γ -composition factors of the group \mathfrak{G} . An important problem is the study of the internal properties of the radical. Our immediate aim will be to consider the relation between the radical $a(\Gamma, \mathfrak{G})$ and the locally stable radical of the group Γ under certain restrictions. In the general case, as is not hard to show, these radicals are not incident.

Recall the definition of the locally stable radical ⁽¹⁾. A group Γ is called **stable** if in \mathfrak{G} there is an ascending normal series of Γ -admissible subgroups, in all factors of which the elements of Γ induce the identity automorphisms. Γ is locally stable if, for every subgroup Σ of Γ with a finite number of generators in \mathfrak{G} , there is a local system of Σ -admissible subgroups in which Σ acts as a stable group. We note here that, in the case when \mathfrak{G} is a soluble group, the stated property of the group Σ is equivalent to the stability of Σ with respect to the whole group \mathfrak{G} ⁽¹⁾. The locally stable radical of the group Γ is the normal divisor in Γ generated by all locally stable normal divisors of Γ . For the locally stable radical we shall use here the notation $\beta(\Gamma, \mathfrak{G})$.

Let Φ be a subgroup in Γ . We shall say that for the sequence $\mathfrak{G}, \Gamma, \Phi$ **condition

(*) is satisfied **if in the group \mathfrak{G} there is a normal system of Γ -admissible subgroups, in all factors of which the elements of Φ induce the identity automorphisms. If, moreover, Φ coincides with Γ , then Γ is called a weakly stable group****. We record the simplest cases in which condition (*) is satisfied.

1. If the locally nilpotent radical of the group \mathfrak{G} has finite special rank, then condition (*) is satisfied for the sequence $\mathfrak{G}, \Gamma, \beta(\Gamma, \mathfrak{G})$.
2. If Φ is an externally nilpotent normal divisor in Γ , then condition (*) is satisfied for the sequence $\mathfrak{G}, \Gamma, \Phi$ ⁽¹⁾.

Recall that Φ is called an **externally nilpotent group** if in \mathfrak{G} there is a finite series stable with respect to Φ . Using the well-known method of A. I. Mal'cev ⁽²⁾, it is not hard to show that condition (*) for a fixed sequence $\mathfrak{G}, \Gamma, \Phi$ can be written in the language of the narrow predicate calculus. As the underlying set $M = (a, b, c, \dots)$ one takes the set playing the role of indices of the terms of a suitable normal system. In this set two individual elements a_0 and a_1 must be fixed. The basic predicates defined on the set M are the order relation and all possible predicates of the form $A_g(a)$, where the indices g run through the whole group \mathfrak{G} , meaning, in content, that the element g belongs to the subgroup with index a . The first group of axioms consists of axioms asserting that M is an ordered set. Then come axioms ensuring normality of the supplemented system. To these two groups of axioms one must add the following groups of axioms:

$$(a) (A_g(a) \rightarrow A_{g\sigma}(a)), \quad (Ea) (\bar{A}_g(a) \& \bar{A}_{g^{-1}g\sigma}(a)).$$

Here g ranges over the whole group \mathfrak{G} , while σ in the first case ranges over the whole group Γ , and in the second case over the subgroup Φ .

The existence of a model of such a system of axioms means that the required normal system exists in the group \mathfrak{G} .

Relying on the indicated possibility of characterizing condition (*), one easily obtains the following proposition.

Theorem 1. *Let a representation f of the group Γ by automorphisms of the group \mathfrak{G} be given, and let Φ be some subgroup in Γ . Let also H and Σ be arbitrary subgroups with a finite number of generators in \mathfrak{G} and Γ , respectively. Then, if all sequences of the form $H^\Sigma, \Sigma, \Sigma \cap \Phi$ satisfy condition (*), then the sequence $\mathfrak{G}, \Gamma, \Phi$ also satisfies this same condition.*

Here, as usual, H^Σ denotes the Σ -closure of the subgroup H , and the representation of Σ with respect to H^Σ induced by the representation f is considered.

Theorem 2. *The inclusion*

$$\beta(\Gamma, \mathfrak{G}) \subset \alpha(\Gamma, \mathfrak{G})$$

holds in the following cases:

- 1) for every Γ -composition pair H, F of the group \mathfrak{G} , the locally nilpotent radical has finite rank in the factor group F/H ;
- 2) all Γ -composition factors F/H are Abelian and Γ is an LM-group;
- 3) the representation f is algebraic.

The proof of the theorem is not difficult to obtain from the preceding considerations. We note only that a representation is called **algebraic** if, for every element $g \in \mathfrak{G}$ and every finitely generated subgroup Σ of Γ , the subgroup g^Σ has a finite number of generators. The representation of any locally finite group is algebraic. Hence, in particular, it follows that if the group Γ is locally finite, then $\beta(\Gamma, \mathfrak{G}) \subset \alpha(\Gamma, \mathfrak{G})$.

Theorem 3. *If the groups \mathfrak{G} and Γ are locally finite, then the radicals $\alpha(\Gamma, \mathfrak{G})$ and $\beta(\Gamma, \mathfrak{G})$ coincide. In the case of a faithful representation, moreover, the radical of the representation is an abstract locally nilpotent group.*

Here the inclusion $\alpha(\Gamma, \mathfrak{G}) \subset \beta(\Gamma, \mathfrak{G})$ follows from the following remark: if the representation f is algebraic, the group \mathfrak{G} is locally finite, and Γ is weakly stable with respect to \mathfrak{G} , then Γ is locally stable.

The second assertion of the theorem follows from the first and from (3). All the preceding arguments naturally carry over to the case where the group \mathfrak{G} is considered together with some domain Ω of distributive operators.

It is clear that in this case the subgroup is an Ω -admissible subgroup, and the automorphism is an operator automorphism. Let us note, in particular, that the following analogue of Theorem 3 holds:

Theorem 4. *Let a representation of a locally finite group Γ with respect to a linear space \mathfrak{G} be given. Then*

$$\alpha(\Gamma, \mathfrak{G}) = \beta(\Gamma, \mathfrak{G}).$$

If the representation is faithful, then $\alpha(\Gamma, \mathfrak{G})$ is a locally nilpotent group.

It is easy to observe that the radical $\alpha(\Gamma, \mathfrak{G})$ is, in a known sense, an analogue of the Jacobson radical from ring theory (4). Let us define one analogue of the Levitzki radical.

As is known, if K is the ring of endomorphisms of an abelian group \mathfrak{G} , the external and internal local nilpotence of such a ring are equivalent notions. Recall that K is called *externally nilpotent* if in the group \mathfrak{G} there is a series of K -admissible subgroups \mathfrak{G}_i , $\mathfrak{G}_i \supset \mathfrak{G}_{i+1}$, $\mathfrak{G}_0 = \mathfrak{G}$, $\mathfrak{G}_n = 0$, such that $\mathfrak{G}_i a \subset \mathfrak{G}_{i+1}$ for every i and for every $a \in K$. K is *externally locally nilpotent* if every subring of K with a finite number of generators is externally nilpotent.

Let now a representation of a group Γ with respect to a group \mathfrak{G} (not necessarily commutative) be given. By analogy with rings, the group Γ will be called

externally locally nilpotent if every subgroup of Γ having a finite number of generators is nilpotent with respect to \mathfrak{G} . In the case of a faithful representation, external local nilpotence of Γ implies its internal local nilpotence^(5,6). The converse assertion, unlike for rings, is false. Denote by $\gamma(\Gamma, \mathfrak{G})$ the normal divisor of the group Γ generated by all externally locally nilpotent normal divisors of Γ . The analogy between $\gamma(\Gamma, \mathfrak{G})$ and the Levitzki radical of the ring of endomorphisms is evident.

Theorem 5. *The radical $\gamma(\Gamma, \mathfrak{G})$ is an externally locally nilpotent group.*

The proof of this theorem can be obtained by the methods of work⁽¹⁾, taking into account that $\gamma(\Gamma, \mathfrak{G})$ is contained in the external radical of the group Γ , defined in⁽⁷⁾.

From the definitions the inclusion follows immediately

$$\gamma(\Gamma, \mathfrak{G}) \subset \beta(\Gamma, \mathfrak{G}).$$

Under certain restrictions these two radicals coincide.

Theorem 6. *If Γ is an LM-group, then the radical $\gamma(\Gamma, \mathfrak{G})$ belongs to the radical $\alpha(\Gamma, \mathfrak{G})$.*

We shall next consider the case when there is a direct connection between the radical $\gamma(\Gamma, \mathfrak{G})$ of the group of automorphisms and the Levitzki radical of the endomorphism ring.

Let \mathfrak{G} be an abelian group, K some ring of endomorphisms of the group \mathfrak{G} , containing the identity e , which is the identity operator, and let Γ be the totality of all invertible elements of K . Γ is a group of automorphisms of the group \mathfrak{G} . Denote by $L(K)$ the Levitzki radical of the ring K .

The following inclusion is easily verified:

$$e + L(K) \subset \gamma(\Gamma, \mathfrak{G}).$$

Under certain additional restrictions this inclusion can be replaced by equality. This will be the case, for example, if Γ generates the whole ring K and, in addition, one of the following conditions is satisfied: a) the group $\gamma(\Gamma, \mathfrak{G})$ is externally nilpotent; b) Γ is an LM-group.

The question of the exact conditions for the coincidence of $e + L(K)$ and $\gamma(\Gamma, \mathfrak{G})$ remains unresolved for the time being.

Let us also note that in the situation under consideration there is likewise the inclusion

$$e + D(K) \subset \alpha(\Gamma, \mathfrak{G}).$$

Here by $D(K)$ we have denoted the Jacobson radical of the ring K .

We now turn to some applications for the special case where the group Γ is taken to be the group \mathfrak{G} of inner automorphisms of the group \mathfrak{G} . First of all,

let us agree to denote by $\alpha(\mathfrak{G})$, $\beta(\mathfrak{G})$, and $\gamma(\mathfrak{G})$, respectively, the full inverse images in \mathfrak{G} of the radicals $\alpha(\mathfrak{G}, \mathfrak{G})$, $\beta(\mathfrak{G}, \mathfrak{G})$, and $\gamma(\mathfrak{G}, \mathfrak{G})$. It has already been noted that $\beta(\mathfrak{G})$ coincides with the locally nilpotent radical of the group \mathfrak{G} .

From Theorem 3 and the remark concerning the radical $\beta(\mathfrak{G})$, the following follows immediately.

Theorem 7. *In a locally finite group the locally nilpotent radical coincides with the radical $\alpha(\mathfrak{G})$.*

This theorem shows, in particular, that in locally finite groups the factor group by the locally nilpotent radical has a good property: it is a subdirect product of automorphism groups induced by the group on the factors of its chief systems. Theorem 7 is known for finite groups. In a recent paper of P. Hall ⁽⁸⁾ an analogous theorem is proved for some other classes of groups.

From Theorem 2 for *LM*-groups we obtain the inclusion $\beta(\mathfrak{G}) \subset \alpha(\mathfrak{G})$. The same inclusion also holds in the case where the group \mathfrak{G} is an extension of a *ZA*-group by means of an *LM*-group. Indeed, it is easy to show that in this case, for any subgroup $H \subset \mathfrak{G}$ having a finite number of generators, the intersection $H \cap \beta(\mathfrak{G})$ is a *ZA*-group. Our assertion now follows immediately from the local theorem.

Concerning the radical $\gamma(\mathfrak{G})$, let us note that it contains the Fitting radical, i.e., the subgroup generated by all nilpotent normal divisors of the group. Probably $\gamma(\mathfrak{G})$ is wider than the Fitting radical.

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Note: Figure translations are in progress. See original paper for figures.

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