MATHEMATICS

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Abstract

Full Text

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BROWNIAN MOTION WITH KILLING MEASURE μ AND SPEED MEASURE ν

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1. Let E be l-dimensional Euclidean space, and let G be its open subset. We shall construct on G a family of homogeneous continuous Markov processes X_ν^μ, each of which is characterized by two measures μ and ν on the set G. The process X_ν^μ is called Brownian motion with killing measure μ and speed measure ν. If sufficiently smooth derivatives $d\mu/d\nu$ and $d\nu/d\lambda$ exist (λ is Lebesgue measure), then the process X_ν^μ is a diffusion process with generating differential operator $\mathfrak{D}f(x) = \frac{1}{2}a(x)\Delta f(x) - c(x)f(x)$, where Δ is the Laplace operator, $a = [d\nu/d\lambda]^{-1}$, and $c = d\mu/d\nu$. In particular, for $G = E$, X_λ^0 is the Wiener process. For $G = E$, the processes X_ν^0 were considered in ([1]).

2. Let $X = (x_t, +\infty, \mathcal{M}_t, \mathbf{P}_x)$ be the Wiener process $*$, i.e. a continuous homogeneous Markov process in the space E with transition density $p(t, x, y) = (2\pi t)^{-l/2} \exp[-|y-x|^2/2]$ $**$. We agree to denote by $\tau(\Gamma)$ the moment of the first exit of the process X from the set Γ. Put $U \in \mathcal{E}_0$ if for every $x \in U$ there exists a set Γ such that $x \in \Gamma \subseteq U$ and $\mathbf{P}_x\{\tau(\Gamma) > 0\} = 1$. The system $\mathcal{E}_0$ defines in E the natural topology (cf. ([3,4])).

Put $w_1(x) = 1$, $w_2(x) = \max(-\ln|x|, 1)$, $w_l(x) = |x|^{2-l}$ for $l \geq 3$. A measure μ in the space E $***$ will be called a W-measure if

$$\sup_{x \in E} \int_{|y-x| \leq 1} w(y-x)\mu(dy) < \infty.$$

A measure μ given on an open set G will be called an S-measure if there exists a sequence of closed sets $\Gamma_n \uparrow G$ such that each measure

$$\mu_n(\Gamma) = \mu(\Gamma \cap \Gamma_n) \qquad (\Gamma \in \mathcal{B}_E)$$

is a W-measure and $\tau(\Gamma_n) \uparrow \tau(G)$ (a.s.) $****$.

Slightly generalizing the result of ([5]), it is not difficult to show that to every S-measure μ on the open set G there corresponds a homogeneous continuous nonnegative additive functional

$$\varphi_t^s(\mu) = \int_s^t \frac{d\mu}{d\lambda}(x_u)\, du \qquad (0 \le s \le t < \tau(G))$$

of the Wiener process X (the integral on the right has an understandable meaning if the derivative $d\mu/d\lambda$ exists; in the general case this integr–

$*$ We use the terminology and notation of the book ([2]).

$**$ By $|x|$ is denoted the Euclidean length of the vector x.

$***$) Only measures defined on Borel sets are considered. We shall denote by $\mathcal{B}_G$ the system of all Borel subsets of the set G.

($****$) The notation (a.s.) means that the corresponding property holds almost surely with respect to all measures $\mathbf{P}_x$ $(x \in E)$.

is determined by means of a certain limiting transition). We shall say that the measure μ is positive if $\mu(U) > 0$ for every nonempty set $U \in \mathcal{C}_0$. According to (1), if the measure μ is positive, then the functional $\varphi(\mu)$ is positive, i.e., for $0 \le s < t$, $\varphi_t^s(\mu) > 0$ for almost all ω in the set $\{\tau(G) > t\}$.

3. Extend the functional $\varphi(\mu)$ to the interval $[0, \infty)$, putting $\varphi_t^s(\mu) = \infty$ for $t \ge \tau(G)$. According to (2), to the extended functional there corresponds a certain subprocess X^μ of the process X. It is obtained from X if each trajectory of X is cut off at the random time $\tilde{\xi}$; the conditional probability of the inequality $\tilde{\xi} > t$, for a fixed trajectory of the process X, is equal to $\exp[-\varphi_t(\mu)]$. Obviously, $\tilde{\xi} \le \tau(G)$ (a.s.). Therefore it is natural to consider the process X^μ only on the set G.

Let ν be some positive S-measure. It is not hard to verify that the formula $\tilde{\varphi}_t^s = \varphi_t^s(\nu)$ $(0 \le s \le t < \tilde{\xi})$ defines a homogeneous continuous positive additive functional of X^μ. The function $\tilde{\varphi}_t$ carries out a topological mapping of the interval $[0, \tilde{\xi})$ onto the interval $[0, \hat{\xi})$, where $\hat{\xi} = \varphi_{\tilde{\xi}}$. Denote by β_t the inverse mapping of $[0, \hat{\xi})$ onto $[0, \tilde{\xi})$. The formula $\hat{x}_t = \tilde{x}_{\beta_t}$ $(0 \le t < \hat{\xi})$ defines a random change of time in the process X^μ, corresponding to the functional $\tilde{\varphi}$. According to (6), as a result of this change, from X^μ there is obtained a certain continuous Markov process X_ν^μ on the set G. We shall call it **Brownian motion with killing measure μ and speed measure ν**. Note that $X_\lambda^\mu = X^\mu$.

4. Let us express the elements $(\hat{x}_t, \hat{\xi}, \hat{\mathcal{M}}_t, \hat{\mathbf{P}}_x)$ of the Brownian motion X_ν^μ in terms of the basic elements of the Wiener process X. The space of elementary events for X_ν^μ is $\hat{\Omega} = \Omega \times [0, \infty]$, where Ω is the space of elementary events for X. Put $\tau = \tau(G)$. The killing time and trajectory of X_ν^μ are expressed by the formulas

$$\hat{\xi}(\omega, u) = \varphi_{\min[\tau(\omega), u]}(\nu, \omega),$$

$$\hat{x}_t(\omega, u) = x_{\beta_t(\omega)}(\omega) \quad \text{for } t < \hat{\xi}(\omega, u).$$

The σ-algebra $\hat{\mathcal{M}}_t$ consists of all subsets of the space $\hat{\Omega}$ having the form $\{A, \hat{\xi} > t\} = \{A, \varphi_\tau > t\} \times (t, \infty]$, where $A \in \mathcal{M}_{\beta_t}$.* The domain of definition $\hat{\mathcal{M}}^0$ of the measures $\hat{\mathbf{P}}_x$ is expressed through the domain of definition $\mathcal{M}^0$ of the measures $\mathbf{P}_x$ by the formula $\hat{\mathcal{M}}^0 = \mathcal{M}^0 \times \mathcal{B}_{[0,\infty]}$. The mathematical expectation corresponding to the measure $\hat{\mathbf{P}}_x$ is given by the formula

$$\hat{\mathbf{M}}_x \eta = \int_\Omega \left[\int_0^{\tau(\omega)} \eta(\omega, u) e^{-\varphi_u(\mu)} \varphi(\mu, du) + e^{-\varphi_\tau(\mu)} \eta(\omega, \tau) \right] \mathbf{P}_x(d\omega)$$

(to the monotone function $\varphi_t(\mu)$ there corresponds a certain measure on the half-line $[0, \infty)$; integration inside the square brackets is with respect to this measure).

From the formula for $\hat{\mathbf{M}}_x \eta$ it follows easily that the semigroup of operators corresponding to the process X_ν^μ is determined by the formula

$$\hat{T}_t f(x) = \hat{\mathbf{M}}_x f(\hat{x}_t) = \mathbf{M}_x \alpha_{\beta_t} f(x_{\beta_t}) \chi_{\beta_t < \tau},$$

where $\alpha_u = \exp[-\varphi_u(\mu)]$. If $\tau(\Gamma)$ and $\hat{\tau}(\Gamma)$ are the times of first exit from some set $\Gamma \subseteq G$, respectively for the processes X and X_ν^μ, then

* In accordance with (2), we put $A \in \mathcal{M}_{\beta_t}$ if $\{A, \beta_t \le u\} \in \mathcal{M}_u$ for every $u > 0$.

$$\tau(\Gamma) = \min[\varphi_{\tau(\Gamma)}, \hat{\zeta}], \quad \hat{x}_{\hat{\tau}(\Gamma)} = x_{\tau(\Gamma)} \quad \text{when } \hat{\tau}(\Gamma) < \hat{\zeta},$$

and for any Borel function f

$$\hat{\mathbf{M}}_x \int_0^{\hat{\tau}(\Gamma)} f(\hat{x}_t)\, dt = \mathbf{M}_x \int_0^{\tau(\Gamma)} \alpha_u f(x_u)\, \varphi(\nu, du),$$

$$\hat{\mathbf{M}}_x f[\hat{x}_{\hat{\tau}(\Gamma)}] = \mathbf{M}_x \alpha_{\tau(\Gamma)} f[x_{\tau(\Gamma)}]\, \chi_{\tau > \tau(\Gamma)}.$$

From these formulas, in particular, it is clear that for every U

$$\hat{\mathbf{P}}_x\{\hat{x}_{\hat{\tau}(U)} \in \Gamma\} \le \mathbf{P}_x\{x_{\tau(U)} \in \Gamma\},$$

i.e. the exit probabilities for the process X_ν^μ are majorized by the corresponding probabilities for the Wiener process. It would be very interesting to describe the class of all normal processes possessing this property. It is natural to think that this class, if it does not coincide with the class of all processes X_ν^μ, is at any rate very close to it*. (In the one-dimensional case the two classes coincide.) It is proved in (1) that, for $G = E$, the class of processes having the same exit probabilities as the Wiener process coincides with the class of processes X_ν^0.

5. Denote by $\mathcal{U}(G)$ the collection of open sets U whose closures are compact and contained in G. Denote by $\mathcal{K}(G)$ the collection of all infinitely differentiable functions on G, each of which is equal to zero outside some $U \in \mathcal{U}(G)$. Let f be a locally integrable function on G, and let ψ be a countably additive locally finite function on $\mathcal{B}_G$. If for every $F \in \mathcal{K}(G)$ the equality

$$\int_G F(y)\,\psi(dy) = -\frac{1}{2}\int_G \Delta F(y) f(y)\,dy$$

is fulfilled, then we shall say that f belongs to the domain of definition $D_\Psi(G)$ of the mapping Ψ, and shall write $\psi = \Psi f$.

Let μ be an arbitrary measure on G. Put $f \in D_\mu(G)$ and $\gamma = \mu f$, if f is locally integrable with respect to μ and

$$\gamma(\Gamma) = \int_\Gamma f(x)\,\mu(dx) \qquad (\Gamma \in \mathcal{B}_G).$$

Theorem 1. *The set of all functions harmonic for the Brownian motion X_ν^μ coincides with the set of all $\mathcal{C}_0$-continuous solutions of the equation $\Psi f + \mu f = 0$.*

Let μ and ν be measures given in some neighborhood of a point x. The linear local operator $\mathfrak{D} = -D_\nu(\Psi + \mu)$ is defined as follows. Let f be a $\mathcal{C}_0$-continuous function defined in a neighborhood of x, and suppose there is a neighborhood U of the point x such that $f \in D_\Psi(U) \cap D_\mu(U)$ and the countably additive function $-\Psi f - \mu f$ has, with respect to the measure ν, a $\mathcal{C}_0$-continuous derivative F on U satisfying the condition

$$\int_U g(x,y)|F(y)|\,\nu(dy) < \infty \qquad (x \in U)$$

($g(x,y)$ is the Green function of the Laplace operator on U). Then we put $f \in D_\mathfrak{D}(x)$ and $\mathfrak{D} f(x) = F(x)$.

The characteristic operator of an arbitrary process X in the topology $\mathcal{C}_0$ is defined by the equality

$$\mathfrak{A} f(x) = \lim_{U \downarrow x} \frac{\mathbf{M}_x f[x_{\tau(U)}] - f(x)}{\mathbf{M}_x \tau(U)},$$

where U is a $\mathcal{C}_0$-neighborhood of the point x, and $\tau(U)$ is the moment of first exit of X from U.

* Note added in proof. This hypothesis has recently been proved by M. G. Shur.

Theorem 2. The characteristic operator $\mathfrak{A}$ of the Brownian motion X^μ_ν in the topology $\mathcal{C}_0$ is an extension of the operator $\mathfrak{D} = -D_\nu(\Psi + \mu)$.

Denote by $\mathfrak{D}_0$ the operator defined by the equality

$$\mathfrak{D}_0 f(x) = \mathfrak{D} f(x) = -D_\nu(\Psi + \mu) f(x) \qquad (x \in G)$$

on the set of all functions f satisfying the conditions: a) $f \in D_{\mathfrak{D}}(x)$ for all $x \in G$; b) f and $\mathfrak{D} f$ are bounded Borel functions; c) $\lim_{t \uparrow \tau} f(x_t) = 0$ (a.s. $\{\varphi_\tau(\nu) + \varphi_\tau(\mu) < \infty\}$) ($x_t$ is a trajectory of the Wiener process).

Theorem 3. If the restriction of the measure ν to each set $U \in \mathfrak{U}(G)$ is a W-measure, then the weak infinitesimal operator* of the process X^μ_ν coincides with $\mathfrak{D}_0$.

Theorem 4. The semigroup $\widehat{T}_t$ corresponding to an arbitrary Brownian motion X^μ_ν maps into itself the space C^0 of all bounded $\mathcal{C}_0$-continuous functions. The weak infinitesimal operator of the semigroup $\widehat{T}_t$ in the space C^0 coincides with the closure of $\mathfrak{D}_0$ (with respect to weak convergence).

In a number of cases the condition c) entering into the definition of $\mathfrak{D}_0$ can be simplified. For example, if $(r_1, r_2) = (-\infty, +\infty)$, then condition c) may simply be omitted.

6. By a regular process in the interval (r_1, r_2) we shall mean a continuous strong Markov process satisfying the following "complete accessibility" condition: for any a and b from (r_1, r_2) the probability, starting from a, of ever visiting b is positive.

It can be proved that the functions harmonic for a regular process $\widetilde{X}$ have locally bounded variation and form a two-dimensional linear space. Let h_1 and h_2 be some basis of this space. We shall call the expression $h_1\,dh_2 - h_2\,dh_1$ the characteristic differential form of the process $\widetilde{X}$. Integrating this form, one can construct a function $y(x)$ such that $h_1\,dh_2 - h_2\,dh_1 = dy$. The function $y(x)$ is strictly monotone. We shall call it the **canonical coordinate** for the process $\widetilde{X}$. By a monotone continuous transformation of the phase space one can arrange that $y(x) \equiv x$. If this condition is fulfilled, then we say that X is a process without drift.

Theorem 5. The class of all regular processes without drift in the open interval (r_1, r_2) coincides with the class of all Brownian motions X^μ_ν in this interval.

We note that in the one-dimensional case the class of S-measures on the interval (r_1, r_2) coincides with the class of all measures finite on each segment $[a, b] \subset (r_1, r_2)$; every S-measure satisfies the conditions of Theorem 3. Therefore the weak infinitesimal operator is described by Theorem 3.

We also note that every regular process on the closed segment $[r_1, r_2]$ can be obtained from some Brownian motion on the line $(-\infty, +\infty)$ by reflection at the points r_1 and r_2.

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* For the definition of the weak infinitesimal operator, see, for example, (7).

Note: Figure translations are in progress. See original paper for figures.