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Abstract

Full Text

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EXPANSION IN EIGENFUNCTIONS OF ORDINARY LINEAR DIFFERENTIAL OPERATORS ON A FINITE INTERVAL

(Presented by Academician I. G. Petrovsky on 7 V 1962)

Let us consider on the interval $[0, 1]$ the boundary-value problem defined by the differential equation

$$y^{(n)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y + \lambda y = 0 \tag{1}$$

and by the normalized ⁽¹⁾, p. 51) boundary conditions:

$$U_i(y) = \sum_{j=0}^{\sigma_i} a_{in-\sigma_i+j} y^{(\sigma_i-j)}(0) + \sum_{j=0}^{\sigma_i} b_{in-\sigma_i+j} y^{(\sigma_i-j)}(1) = 0;$$

$$|a_{in-\sigma_i+j}| + |b_{in-\sigma_i+j}| > 0, \quad i = 1, 2, \dots, n;$$

$$n-1 \geq \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0, \quad \sigma_k > \sigma_{k+2}.$$
(2)

We assume that n is odd; $p_i(x)$ ($i = 2, \dots, n$) are certain complex-valued functions, with $p_i^{(n-i)}(x) \in \mathcal{L}[0, 1]$ ($i = 2, \dots, n$); $U_i(y)$ ($i = 1, 2, \dots, n$) are linearly independent linear forms with complex coefficients in $y(0), \dots, y^{(n-1)}(0), y(1), \dots, y^{(n-1)}(1)$; λ is a complex parameter. Questions concerning expansion in eigenfunctions of such boundary-value problems were considered by G. D. Birkhoff ⁽³⁾, J. D. Tamarkin ⁽⁵⁾, and M. Stone ⁽⁴⁾. In this, additional requirements, called regularity conditions ⁽¹⁾, p. 51), were imposed on the boundary conditions. Boundary-value problems with irregular boundary conditions were studied by L. Ward ⁽⁶⁾ and M. V. Keldysh ⁽²⁾. Below, some other classes of boundary-value problems with irregular boundary conditions are considered.

Let $\lambda = \rho^n$, $\omega_k = e^{\frac{2k-1}{n}\pi i}$ ($k = 1, \dots, n$), and let $y_1(x), y_2(x), \dots, y_n(x)$ be a fundamental system of solutions of equation (1), defined in ⁽¹⁾, pp. 45-46. Then the equation for the eigenvalues is represented in the form

$$\begin{aligned}
 \Delta(\lambda) &= \begin{vmatrix} U_{11} \dots U_{1n} \\ \dots \dots \dots \\ U_{n1} \dots U_{nn} \end{vmatrix} = \begin{vmatrix} A_{11} + B_{11}e^{\rho\omega_1} \dots A_{1n} + B_{1n}e^{\rho\omega_n} \\ \dots \dots \dots \\ A_{n1} + B_{n1}e^{\rho\omega_1} \dots A_{nn} + B_{nn}e^{\rho\omega_n} \end{vmatrix} = \\
 &= \begin{vmatrix} A_{11} \dots A_{1n} \\ \dots \dots \dots \\ A_{n1} \dots A_{nn} \end{vmatrix} + \sum (\pm) \begin{vmatrix} A_{1i_1} \dots A_{1i_{n-1}} B_{1i_n} \\ \dots \dots \dots \\ A_{ni_1} \dots A_{ni_{n-1}} B_{ni_n} \end{vmatrix} e^{\rho\omega_{i_n}} + \dots \quad (3) \\
 &\dots + \begin{vmatrix} B_{11} \dots B_{1n} \\ \dots \dots \dots \\ B_{n1} \dots B_{nn} \end{vmatrix} e^{\rho(\omega_1 + \dots + \omega_n)} = 0,
 \end{aligned}$$

where $U_i(y_j) = U_{ij} = A_{ij} + B_{ij}e^{\rho\omega_j}$; A_{ij} (respectively $B_{ij}e^{\rho\omega_j}$) is the part of the form $U_i(y_j)$ at the point 0 (respectively at the point 1).

Introduce the following notation. In the case $0 \leq \arg \rho \leq \pi/n$, denote by p_0 and p_1 the coefficients in relation (3)', respectively, at

$$\exp \left[\rho \left(\sum_{i=1}^{\mu} \omega_i + \sum_{i=3\mu+2}^n \omega_i \right) \right] \quad \text{and} \quad \exp \left[\rho \sum_{i=1}^{\mu} \omega_i + \sum_{i=3\mu+1}^n \omega_i \right]$$

for $n = 4\mu + 1$ (or at

$$\exp \left[\rho \left(\sum_{i=1}^{\mu+1} \omega_i + \sum_{i=3(\mu+1)}^n \omega_i \right) \right] \quad \text{and} \quad \exp \left[\rho \sum_{i=1}^{\mu} \omega_i + \sum_{i=3(\mu+1)}^n \omega_i \right]$$

for $n = 4\mu + 3$); in the case $\pi/n \leq \arg \rho \leq 2\pi/n$, denote by \tilde{p}_0 and \tilde{p}_1 the coefficients at

$$\exp \left[\rho \left(\sum_{i=1}^{\mu-1} \omega_i + \sum_{i=3\mu+1}^n \omega_i \right) \right] \quad \text{and} \quad \exp \left[\rho \left(\sum_{i=1}^{\mu} \omega_i + \sum_{i=3\mu+1}^n \omega_i \right) \right]$$

for $n = 4\mu + 1$ (or at

$$\exp \left[\rho \left(\sum_{i=1}^{\mu} \omega_i + \sum_{i=3\mu+2}^n \omega_i \right) \right] \quad \text{and} \quad \exp \left[\rho \left(\sum_{i=1}^{\mu} \omega_i + \sum_{i=3(\mu+1)}^n \omega_i \right) \right]$$

for $n = 4\mu + 3$).

We shall consider the boundary-value problem (1)–(2) under the following additional requirements:

$$\lim_{\rho \rightarrow \infty} \frac{p_0}{\rho^{\alpha_1}} \neq 0, \quad \lim_{\rho \rightarrow \infty} \frac{p_1}{\rho^{\beta_1}} \neq 0, \quad \lim_{\rho \rightarrow \infty} \frac{\tilde{p}_0}{\rho^{\alpha_2}} \neq 0, \quad \lim_{\rho \rightarrow \infty} \frac{\tilde{p}_1}{\rho^{\beta_2}} \neq 0, \quad (4)$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are certain real numbers. These conditions generalize the conditions of regularity. It is not difficult to show that the boundary conditions are regular if and only if

$$\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \sum_{i=1}^n \sigma_i$$

(with respect to σ_i ($i = 1, 2, \dots, n$), see (2)).

Definition 1. We shall call the boundary-value problem A under conditions (4) a **boundary-value problem of class** $(\alpha_1, \beta_1, \alpha_2, \beta_2)$.

Then the following assertions hold.

Theorem 1. The eigenvalues of the boundary-value problem A of class $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ form two infinite sequences $\{\lambda'_k\}_{k=1}^{\infty}$, $\{\lambda''_k\}_{k=1}^{\infty}$ such that

$$\lambda'_{k+h'} = [(2k-1)\pi]^n [1 + o(1)] e^{i\pi/2},$$

$$\lambda''_{k-h''} = [(2k-1)\pi]^n [1 + o(1)] e^{i3\pi/2},$$

moreover, beginning with some point, all eigenvalues are simple (h', h'' are certain integers depending only on the problem A).

Number the eigenvalues of the boundary-value problem A of class $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ in increasing order of their moduli:

$$|\lambda_1| \leq |\lambda_2| \leq \dots$$

Let $\{C_p\}_{p=1}^{\infty}$ be a sequence of contours in the λ -plane, each of which contains $\lambda_1, \dots, \lambda_p$ and contains no other λ_j ($j = p+1, p+2, \dots$).

Definition 2. We shall call the function

$$S_q(A, f) = \frac{1}{2\pi i} \int_0^1 \int_{C_p} G(x, \xi; \lambda) d\lambda f(\xi) d\xi$$

the **partial sum of the Fourier series in eigenfunctions and associated functions** ((1), pp. 21–24) of the boundary-value problem A for a certain integrable function $f(x)$ ($G(x, \xi; \lambda)$ is the resolvent (Green's function)); q is

the number of all eigenfunctions and associated functions corresponding to the eigenvalues $\lambda_1, \dots, \lambda_p$ that have fallen inside C_p).

This definition will be justified if one evaluates the integral

$$\frac{1}{2\pi i} \int_0^1 \int_{C_p} G(x, \xi; \lambda) d\lambda f(\xi) d\xi$$

by the residue theorem, using the analytic behavior of $G(x, \xi; \lambda)$ ((¹), pp. 37-40). From Theorem 1 it follows easily that, for every sufficiently large q , there is a $p = p(q)$ such that $S_q(A, f) =$

$$= \frac{1}{2\pi i} \int_0^1 \int_{C_{p(q)}} G(x, \xi; \lambda) d\lambda f(\xi) d\xi.$$

Theorem 2. If A is a boundary-value problem of class $(\alpha_1, \beta_1, \alpha_2, \beta_2)$, then

$$\lim_{q \rightarrow \infty} \max_{0 \leq x \leq 1} |f(x) - S_q(A, f)| = 0$$

for every function $f(x) \in D_{A^r}$, where

$$r = \frac{1}{n} \left[\sum_{i=1}^n \sigma_i - \min\{\alpha_1, \beta_1, \alpha_2, \beta_2\} - n + 1 \right] + 2$$

(D_A is the domain of definition of the operator A).

In general, the exponent r cannot be lowered. This theorem generalizes the well-known theorem of Birkhoff ((¹), p. 72) for boundary-value problems with regular boundary conditions.

Theorem 3. The system of eigenfunctions and associated functions of the boundary-value problem A of class $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ is complete in $\mathfrak{L}[0, 1]$.

Theorem 4. Let A, A' be boundary-value problems of classes $(\alpha_1, \beta_1, \alpha_2, \beta_2)$, $(\alpha'_1, \beta'_1, \alpha'_2, \beta'_2)$;

$$\chi \geq \max \left\{ \sum_{i=1}^n \sigma_i - \min_i \{\alpha_i, \beta_i\}; \sum_{i=1}^n \sigma'_i - \min_i \{\alpha'_i, \beta'_i\} \right\}.$$

Let $\{\Gamma_l\}_{l=1}^{\infty}$ be an arbitrary sequence of circles in the λ -plane with common center at the origin, with radii R_l ($l = 1, 2, \dots$) increasing without bound, and such that

$$\inf_{\lambda \in \bigcup_{l=1}^{\infty} \Gamma_l, \lambda_i \in \Lambda} |\lambda - \lambda_i| > 0,$$

where Λ is the set of all eigenvalues of the problems A and A' .

Then, for every function $f(x) \in \mathfrak{L}[0, 1]$,

$$\lim_{l \rightarrow \infty} \max_{0 < \delta \leq x \leq 1 - \delta} \left| \int_0^1 \int_{\Gamma_l} \left(1 - \frac{\lambda^4}{R_l^4}\right)^x [G(x, \xi; \lambda) - G'(x, \xi; \lambda)] d\lambda f(\xi) d\xi \right| = 0,$$

$$\lim_{l \rightarrow \infty} \max_{0 \leq x \leq 1} \left| \int_{\delta}^{1-\delta} \int_{\Gamma_l} \left(1 - \frac{\lambda^4}{R_l^4}\right)^x [G(x, \xi; \lambda) - G'(x, \xi; \lambda)] d\lambda f(\xi) d\xi \right| = 0$$

(δ is an arbitrary number subject to the inequalities $0 < \delta < 1 - \delta$).

This theorem generalizes Stone's theorem (4) on equisummability by Riesz means for regular boundary conditions.

Theorem 5. If A and A' are two boundary-value problems of classes $(\alpha, \alpha, \beta, \beta)$ and $(\alpha', \alpha', \beta', \beta')$, then

$$\lim_{q \rightarrow \infty} \max_{\delta \leq x \leq 1 - \delta} |S_q(A, f) - S_q(A', f)| = 0$$

for every function $f(x) \in \mathfrak{L}[0, 1]$ that vanishes outside $[\delta, 1 - \delta]$ (δ is the same as in Theorem 4).

Theorem 6. Suppose that, for the boundary-value problems A and A' , the following relations hold:

$$p_i = p'_i \left[1 + O\left(\frac{1}{\rho}\right)\right], \quad \tilde{p}_i = \tilde{p}'_i \left[1 + O\left(\frac{1}{\rho}\right)\right] \quad (i = 0, 1).$$

Then there exists an integer h , depending only on the boundary-value problems A and A' , such that for every function $f(x) \in L[0, 1]$ that vanishes outside $[\delta_1, 1 - \delta_2]$:

a)

$$\lim_{q \rightarrow \infty} \max_{0 < x < 1 - \min_i \left\{ \frac{1}{|\alpha_i - \beta_i|} \right\} - \delta_2} |S_q(A, f) - S_{q+h}(A', f)| = 0; \quad (5)$$

if

$$\max_i \{|\alpha_i - \beta_i|\} > 1$$

and $\alpha_i \leq \beta_i$ for $n = 4\mu + 1$ (or $\alpha_i \geq \beta_i$ for $n = 4\mu + 3$), $i = 1, 2$;

b)

$$\lim_{q \rightarrow \infty} \max_{\delta_3 < x < \min_i \left\{ \frac{1}{|\alpha_i - \beta_i|} \right\} + \delta_1 < 1} |S_q(A, f) - S_{q+h}(A', f)| = 0, \quad (6)$$

if

$$\max_i \{ |\alpha_i - \beta_i| \} > 1$$

and $\alpha_i \geq \beta_i$ for $n = 4\mu + 1$ (or $\alpha_i \leq \beta_i$ for $n = 4\mu + 3$), $i = 1, 2$;

c)

$$\lim_{q \rightarrow \infty} \max_{\delta_3 \leq x \leq 1 - \delta_3} |S_q(A, f) - S_{q+h}(A', f)| = 0,$$

if

$$\max_i \{ |\alpha_i - \beta_i| \} \leq 1.$$

Here $\delta_1, \delta_2, \delta_3$ are arbitrary positive numbers satisfying the inequalities appearing in one of the cases a), b), or c).

In general, the intervals of uniform convergence in relations (5) and (6) cannot be extended.

Theorem 7. Suppose that, for the boundary-value problems A and A' generated by the differential expressions

$$y^{(n)} + p_{n-k}(x)y^{(k)} + \dots + p_n(x)y$$

and

$$y^{(n)} + q_{n-k}(x)y^{(k)} + \dots + q_n(x)y,$$

the following relations are satisfied:

$$p_i = p'_i \left[1 + O \left(\frac{1}{\rho^{n-k-1}} \right) \right], \quad \tilde{p}_i = \tilde{p}'_i \left[1 + O \left(\frac{1}{\rho^{n-k-1}} \right) \right], \quad i = 0, 1;$$

$$0 \leq k \leq n - 2 - \max_i \{ |\alpha_i - \beta_i| \}.$$

Then

$$\lim_{q \rightarrow \infty} \max_{0 < \delta \leq x \leq 1 - \delta} |S_q(A, f) - S_{q+h}(A', f)| = 0$$

for every function $f(x) \in L[0, 1]$ that vanishes outside $[\delta, 1 - \delta]$ (h is the same as in Theorem 6).

Remark. Completeness of the eigenfunctions and associated functions in $L[0, 1]$ also holds in the case when each of the coefficients p in

$$\exp \left[\rho \sum \omega_i \right]$$

in relation (3) is either identically equal to zero, or, for it, for some γ ,

$$\lim_{\rho \rightarrow \infty} \frac{p}{\rho^\gamma} \neq 0,$$

and when at least one of the following two alternatives is fulfilled:

$$\text{either } \lim_{\rho \rightarrow \infty} \frac{p_0}{\rho^{\alpha_1}} \neq 0, \quad \lim_{\rho \rightarrow \infty} \frac{\tilde{p}_0}{\rho^{\alpha_2}} \neq 0;$$

$$\text{or } \lim_{\rho \rightarrow \infty} \frac{p}{\rho^{\beta_1}} \neq 0, \quad \lim_{\rho \rightarrow \infty} \frac{\tilde{p}_1}{\rho^{\beta_2}} \neq 0.$$

Similar results are not difficult to obtain for even n .

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Note: Figure translations are in progress. See original paper for figures.

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