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Abstract

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THE INFLUENCE FUNCTION FOR WEAK PERTURBATIONS OF A BAROCLINIC ISOTHERMALLY STRATIFIED ATMOSPHERE

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In the paper by A. S. Monin and A. M. Obukhov ([1]), all solutions of the linearized hydrodynamic equations for an isothermally stratified atmosphere having the form of plane waves were found. There, considerations were also expressed concerning the character of the adaptation of the motion to a stationary geostrophic one. For a more detailed study of the question it is necessary to be able to solve the nonstationary problem with arbitrary initial data and, first of all, to find and investigate the influence function of a point source. The present note is devoted to this problem. Owing to the hyperbolic character of the equations, the influence function is equal to zero outside the cone of characteristics. We decompose it into two parts, corresponding to acoustic and gravitational waves. Since for each of these parts only a part of the spectrum is used, they can no longer be strictly localized in space, i.e., they are different from zero also outside the sound cone; however, there they decrease very rapidly (exponentially) with distance from the cone. In other words, an arbitrary initial perturbation concentrated in a bounded region of space can be decomposed into two parts, exciting pure acoustic and pure gravitational waves, respectively. But each of these parts is concentrated only in "almost" the same bounded region of space. In the quasistatic approximation the group velocity of the waves in the vertical becomes unbounded, and we shall see that the region of influence for each t changes from a sphere into a cylinder with axis parallel to the z-axis. We shall dwell especially on the asymptotics of the influence function in passing from the wave front into the region of influence, i.e., we shall find the "tails" left behind as the front passes. We shall see that the acoustic part decreases very rapidly and that only the gravitational part remains.

1. Let us write out the original system of equations ([1])

$$\begin{gathered}\bar{\rho}u_t = -p_x + l\bar{\rho}v, \qquad \bar{\rho}v_t = -p_y - l\bar{\rho}u, \\ \lambda\bar{\rho}w_t = -(p_z + gp), \qquad \rho_t = -(\bar{\rho}u_x + \bar{\rho}v_y + \bar{\rho}w_z), \\ p_t = -(\kappa - 1)g\bar{\rho}w - c^2(\bar{\rho}u_x + \bar{\rho}v_y + \bar{\rho}w_z), \qquad c^2 = \kappa g H.\end{gathered} \tag{1}$$

The parameter λ, actually equal to unity, has been introduced in order to trace what changes are introduced into our formulas by the assumption of quasistatics ($\lambda = 0$). We introduce new unknowns φ, ψ, χ:

$$\bar{\rho}u = \varphi_x - \psi_y, \qquad \bar{\rho}v = \varphi_y + \psi_x, \qquad \bar{\rho}w = \chi.$$

Then we eliminate ψ, φ, p and ρ:

$$\left(\frac{\partial^2}{\partial t^2} + l^2\right)(c^2\chi_{zz} + \kappa g\chi_z - \lambda\chi_{tt}) + \Delta\left((\kappa-1)g^2\chi + \lambda c^2\chi_{tt}\right) = 0.$$

We note that, leaving an equation for χ, we thereby exclude from consideration "two-dimensional" waves ([1]), since for them $\chi = 0$. Next we make the substitution

$$\chi = e^{-\frac{\kappa g z}{2c^2}}\eta$$

to eliminate the term with the first derivative with respect to height:

$$\left(\frac{\partial^2}{\partial t^2} + l^2\right)\left(c^2\eta_{zz} - \frac{\kappa^2 g^2}{4c^2}\eta - \lambda\eta_{tt}\right) + \Delta\left((\kappa-1)g^2\eta + \lambda c^2\eta_{tt}\right) = 0. \quad (2)$$

It is not difficult to show that the initial conditions, expressed in terms of the initial conditions for the original variables $(\chi, \varphi, \psi, p, \rho)$, are as follows:

$$\eta = \eta_0, \qquad \eta_t = \eta_1, \qquad \eta_{tt} = \eta_2, \qquad \eta_{ttt} = \eta_3, \quad (3)$$

where $\eta_0 = \chi_0$, $\lambda\eta_1 = -\left(\frac{\partial p_0}{\partial z} + g\rho_0\right)$, $\lambda\eta_2 = \left(g + c^2\frac{\partial}{\partial z}\right)\Delta\varphi_0 + \frac{\partial}{\partial z}\left(\chi g + c^2\frac{\partial}{\partial z}\right)\chi_0$,

$$\lambda^2\eta_3 = \lambda\left(g + c^2\frac{\partial}{\partial z}\right)\Delta(l\psi_0 - p_0) - \frac{\partial}{\partial z}\left(\chi g + c^2\frac{\partial}{\partial z}\right)\left(\frac{\partial p_0}{\partial z} + g\rho_0\right).$$

The boundary condition at the Earth' s surface for $z = 0$ is, obviously, $\eta = 0$; at infinity, boundedness of η.

2. We solve equation (2) with the initial conditions (3). To this end we perform a Fourier transform in x and y, and a Laplace transform in t. The resulting equation in z is solved and we pass to the inversion of the transforms. Omitting the intermediate calculations, we give the final expression for the influence function:

$$G=-\frac{1}{8\pi^2 i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\times$$

$$\times\frac{\exp\left\{-\frac{1}{c}\sqrt{\frac{\chi^2 g^2}{4c^2}+\lambda p^2}\sqrt{\delta z^2+\frac{l^2+p^2}{(\chi-1)g^2/c^2+\lambda p^2}(\delta x^2+\delta y^2)}\right\}}{\left(\frac{(\chi-1)g^2}{c^2}+\lambda p^2\right)\sqrt{\delta z^2+\frac{l^2+p^2}{(\chi-1)g^2/c^2+\lambda p^2}(\delta x^2+\delta y^2)}}\,dp; \qquad (4)$$

γ is an arbitrary positive number. If we denote the integral operator with this kernel G by

$$Gu=\int_{-\infty}^{\infty}\!\!\int\!\!\int G(\delta x,\delta y,\delta z,t)\,u(x_1,y_1,z_1)\,dx_1dy_1dz_1$$

$$(\delta x=x-x_1,\ \delta y=y-y_1,\ \delta z=z-z_1),$$

then the solution of our problem (2)–(3) is written in the form

$$\eta=G\left(\Delta_3-\frac{\chi^2g^2}{4c^4}-\frac{\lambda l^2}{c^2}\right)\eta_1+\frac{\partial}{\partial t}G\left(\Delta_3-\frac{\chi^2g^2}{4c^4}-\frac{\lambda l^2}{c^2}\right)\eta_0$$

$$-\frac{\lambda}{c^2}\left(G\eta_3+\frac{\partial}{\partial t}G\eta_2+\frac{\partial^2}{\partial t^2}G\eta_1+\frac{\partial^3}{\partial t^3}G\eta_0\right),\qquad \Delta_3=\lambda\Delta+\frac{\partial^2}{\partial t^2}. \qquad (5)$$

All the functions η are regarded as extended oddly to the region $z<0$. Let us transform integral (4) into a real one. First note that the integrand has the following singular points (branch points):

$$p=\pm im_1,\ \pm im_2,\ \pm im_3,\quad \text{where}\quad m_1=\frac{1}{2}\frac{\chi g}{\sqrt{\lambda}\,c},\quad m_2=\frac{\sqrt{\chi-1}}{\sqrt{\lambda}}\frac{g}{c},\quad m_3=$$

$$=\sqrt{\frac{l^2(\delta x^2+\delta y^2)+[(\chi-1)g^2/c^2]\delta z^2}{\delta x^2+\delta y^2+\lambda\delta z^2}}.$$

It is easy to see that $m_1>m_2>m_3$. For

$$\sqrt{\delta x^2+\delta y^2+\delta z^2}\geq ct$$

(if $\lambda=1$), the function G vanishes, since the contour of integration can be shifted to $+\infty$ ($\gamma\to+\infty$). For large values of t the integrand decreases in the left half-plane. It follows that the contour can be replaced by the one shown in Fig. 1. The entire integral giving G can be split into two parts: the integral over the two outer nonclosed loops and the integral over the inner, closed loops. We denote the first of these by G_{ak} (the acoustic part of the influence function), and the second by G_{gr} (the gravitational part). Let us explain why such names are natural.

$$G_{\text{ak}} = \frac{1}{2\pi^2}\int_{m_1}^{\infty} \sin\tau t\, \frac{\sin\left\{c^{-1}\sqrt{\delta x^2+\delta y^2+\delta z^2}\sqrt{\tau^2-m_1^2}\sqrt{(\tau^2-m_3^2)/(\tau^2-m_2^2)}\right\}}{\sqrt{\lambda}\sqrt{(\tau^2-m_2^2)(\tau^2-m_3^2)}\sqrt{\delta x^2+\delta y^2+\lambda\delta z^2}}\, d\tau, \tag{6}$$

$$G_{\text{gr}} = -\frac{1}{2\pi^2}\int_{m_3}^{m_2} \sin\tau t\, \frac{\cos\left\{c^{-1}\sqrt{\delta x^2+\delta y^2+\delta z^2}\sqrt{m_1^2-\tau^2}\sqrt{(\tau^2-m_3^2)/(m_2^2-\tau^2)}\right\}}{\sqrt{\lambda}\sqrt{(\tau^2-m_3^2)(m_2^2-\tau^2)}\sqrt{\delta x^2+\delta y^2+\lambda\delta z^2}}\, d\tau.$$

Acoustic and gravitational waves were distinguished in paper (1) by their limiting behavior as $\chi \to \infty$ and as $\chi \to 1$. In the first of these limiting transitions, acoustic waves disappeared (the acoustic frequencies went to infinity), while gravitational waves tended to definite finite limits. In the second transition (to indifferent equilibrium), acoustic-

···waves changed insignificantly, while the gravitational frequencies became zero, i.e., the gravitational waves turned into stationary motion. Let us now carry out the same limiting transitions in the expression for our influence function (taking $\lambda = 1$). It is easy to see that, as $\varkappa \to \infty$, we have $G_{\text{ac}} \to 0$, while G_{gr} tends to

$$G_{\text{gr}} = -\frac{1}{2\pi^2}\int_{m_3}^{m_2} \sin\tau t\, \frac{\cos\left\{(2H)^{-1}\sqrt{\delta x^2+\delta y^2+\delta z^2}\,\sqrt{(\tau^2-m_3^2)/(m_2^2-\tau^2)}\right\}}{\sqrt{\delta x^2+\delta y^2+\delta z^2}\,\sqrt{(\tau^2-m_3^2)(m_2^2-\tau^2)}}\, d\tau,$$

where

$$m_2 = \sqrt{\frac{g}{H}}, \qquad m_3 = \sqrt{\frac{l^2(\delta x^2+\delta y^2)+gH^{-1}\delta z^2}{\delta x^2+\delta y^2+\delta z^2}}$$

for all values of t.

Under the second limiting transition, when $\varkappa \to 1$ and $(l \to 0)$:

$$G_{\text{ac}} = \frac{1}{2\pi^2}\int_{\frac{1}{2}\sqrt{g/H}}^{\infty} \sin\tau t\, \frac{\sin\left\{\left[\sqrt{(\delta x^2+\delta y^2+\delta z^2)/gH}\right]\cdot\sqrt{\tau^2-g/4H}\right\}}{\sqrt{\delta x^2+\delta y^2+\delta z^2}\,\tau}\, d\tau.$$

Fig. 1

Figure 1: Fig. 1

At the same time, as is not difficult to show,

$$G_{\text{gr}} = -\frac{1}{4\pi}\frac{1}{\sqrt{\delta x^2+\delta y^2+\delta z^2}}\exp\left\{-\frac{\sqrt{\delta x^2+\delta y^2+\delta z^2}}{2H}\right\},$$

this part of the influence function corresponds to an aperiodic motion. All this makes it possible to regard G_{ac} as the acoustic part and G_{gr} as the gravitational part of the influence function. (When the gravitational frequencies $\pm\omega_{\text{gr}}$ coalesce in pairs, solutions appear that depend linearly on time, as was also obtained in our formulas.)

The function G_{ac} is composed of harmonics $\sin\tau t$ with frequencies τ greater than $m_1 = \frac{1}{2}\sqrt{\varkappa g/H}$; G_{gr} consists of harmonics with frequencies smaller than $m_2 = \sqrt{(\varkappa-1)g/\varkappa H}$. In article (1) it is shown that m_1 is precisely the lower boundary of the acoustic frequencies, while m_2 is the upper boundary of the gravitational frequencies.

Fig. 1

3. Let us now recall the coefficient λ introduced by us and set it equal to zero (the quasistatic approximation). What changes in this case? Integral (4) takes the form

$$G = -\frac{1}{8\pi^2 i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\frac{\exp\left\{-\frac{1}{2H}\sqrt{\delta z^2+\frac{\varkappa H}{(\varkappa-1)g}(l^2+p^2)(\delta x^2+\delta y^2)}\right\}}{\frac{(\varkappa-1)g}{\varkappa H}\sqrt{\delta z^2+\frac{\varkappa H}{(\varkappa-1)g}(l^2+p^2)(\delta x^2+\delta y^2)}}\,dp.$$

This integral can be evaluated. It is equal to

$$G = -\frac{1}{4\pi}\sqrt{\frac{\varkappa H}{(\varkappa-1)g(\delta x^2+\delta y^2)}}\times$$

$$\times\begin{cases} J_0\left(m_3\sqrt{t^2-\frac{\varkappa}{4(\varkappa-1)gH}(\delta x^2+\delta y^2)}\right), & t > \frac{\sqrt{\varkappa}\sqrt{\delta x^2+\delta y^2}}{2\sqrt{(\varkappa-1)gH}},\\ 0, & t < \frac{\sqrt{\varkappa}\sqrt{\delta x^2+\delta y^2}}{2\sqrt{(\varkappa-1)gH}},\end{cases}$$

$$m_3 = \sqrt{\frac{l^2(\delta x^2+\delta y^2)+\frac{(\varkappa-1)g}{\varkappa H}\delta z^2}{\delta x^2+\delta y^2}}.$$

We see that the velocity of propagation of the perturbations in the vertical becomes infinite, while in the horizontal it is equal to $2\sqrt{\chi-1}\sqrt{gH}/\sqrt{\chi} = 2\sqrt{\chi-1}\,c/\chi$, i.e. amounts to $2\sqrt{\chi-1}/\chi \simeq 0.9$ of the former velocity. After this remark we shall henceforth put $\lambda = 1$.

4. Near the leading front the influence function can be calculated only by numerical integration. But, at a distance from the front, outside and inside the cone of influence, one can propose rather simple asymptotic formulas. Let us begin with the asymptotics outside the region of influence. Since here the sum of the acoustic and gravitational parts is zero, it is sufficient to indicate the asymptotics of one of them. We give this formula here without derivation. For $\delta x^2 + \delta y^2 + \delta z^2 \to \infty$ we have

$$G_{\text{ak}} \sim K\,\frac{\exp\left\{-\frac{|\chi-2|}{2\chi H}\sqrt{\delta x^2+\delta y^2+\delta z^2}\right\}}{(\delta x^2+\delta y^2+\delta z^2)^{3/4}}\times$$

$$\times \sin\left\{\sqrt{((\chi-1)g/\chi H - l^2)(\delta x^2+\delta y^2/\chi g H)} + \alpha_0\right\}.$$

K, α_0 are certain constants, whose expression is rather cumbersome. The damping decrement $|\chi-2|/2\chi H$ is $\sqrt{(m_1^2-m_2^2)/\chi g H}$, i.e. it appears at the expense of the "gap" between the acoustic and gravitational frequencies.

We shall also give expressions for the asymptotics of the influence function and of its parts for $t - \sqrt{\delta x^2+\delta y^2+\delta z^2}/\sqrt{\chi g H} \to \infty$; they can be obtained by the saddle-point method:

$$G_{\text{ak}} = \chi^{3/4}H^{1/4}\pi^{-3/2}g^{-5/4}(\chi-2)^{-2}\cos\left(\frac{1}{2}\sqrt{\chi g/H}\,t + \pi/4\right)t^{-3/2},$$

$$G_{\text{gr}} \sim \frac{1}{(2\pi)^{1/2}\sqrt{(m_2^2-m_3^2)(\delta x^2+\delta y^2+\delta z^2)}}\times$$

$$\times\left[\frac{\cos\left(m_2 t - 3(a/2)^{2/3}t^{1/3} + \pi/4\right)}{\sqrt{3m_2 t}} - \frac{\cos(m_3 t + \pi/4)}{\sqrt{m_3 t}}\right],$$

where

$$a = \frac{\sqrt{(m_1^2-m_2^2)(m_2^2-m_3^2)(\delta x^2+\delta y^2+\delta z^2)}}{2m_2\chi g H}.$$

Comparing the two expressions, we note that, whereas the acoustic part decays with time at the rate $t^{-3/2}$, the gravitational part decays only as $t^{-1/2}$, i.e. much more slowly. A short time after the passage of the front, practically only gravitational waves remain. In addition, we see that the acoustic part does not depend on the spatial coordinates, i.e. the whole space moves as one body. As for the gravitational part, it does depend on these coordinates. Moreover, the asymptotics deteriorates as one approaches the straight line $\delta x^2 + \delta y^2 = 0$ (the vertical axis). It begins to take effect later and later, i.e. the damping begins later. On the vertical line itself the asymptotics loses meaning. But here the influence function can be calculated in closed form. It is equal to

$$G_{\mathrm{gr}} = -\frac{1}{4\pi\delta z}\sqrt{\frac{\chi H}{(\chi-1)g}}\exp\left[-\frac{|\chi-2|}{2\chi H}\,\delta z\right]\sin\sqrt{\frac{(\chi-1)g}{\chi H}}\,t.$$

We see that it does not decay with time, but only with height. On the vertical there is always a standing wave.

It would be interesting to carry out a numerical calculation of the influence function in the region where the asymptotic formulas do not apply, and also to compute an example of the evolution of a non-point perturbation.

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CITED LITERATURE

1. A. S. Monin, A. M. Obukhov, *Izv. AN SSSR, ser. geofiz.*, no. 11 (1958).

Note: Figure translations are in progress. See original paper for figures.