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B. PASYNKOV

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Abstract

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B. PASYNKOV

ON THE INTERRELATIONS BETWEEN ZERO-DIMENSIONAL MAPPINGS, UNIVERSAL SPACES, DIMENSION, OPEN ZERO-DIMENSIONAL MAPPINGS, AND INVERSE SPECTRA

(Presented by Academician P. S. Aleksandrov on 12 IV 1962)

Throughout what follows, by a space we mean, unless otherwise specified, a T_1-space, and by a mapping—a continuous mapping. A mapping $f: X \to Y$ is called zero-dimensional in the sense of ind or dim, respectively (which is written as $\operatorname{ind} f = 0$ or $\dim f = 0$), if $\operatorname{ind} f^{-1}(y) \leqslant 0$, or, respectively, $\dim f^{-1}(y) \leqslant 0$, for every point $y \in Y$.

Theorem 1. For every number $n = 0, 1, 2, \ldots$ and every cardinal $\tau \geqslant \mathfrak{c}$, one can construct a bicompactum $P^{n\tau}$ of weight τ possessing the following properties:

1) $\dim P^{n\tau} = \operatorname{ind} P^{n\tau} = \operatorname{Ind} P^{n\tau}$.
2) The bicompactum $P^{n\tau}$ is a universal space* for all n-dimensional, in the sense of dim, spaces of weight τ of the following types: a) bicompacta that are mapped zero-dimensionally onto compacta**; b) metric spaces**; *c) normal spaces possessing a closed and zero-dimensional, in the sense of* ind*, mapping onto a metrizable space with a countable base; d)* $P^{n\tau}$ *is a universal space for all normal spaces of weight* $\leqslant \tau$ *possessing a closed zero-dimensional, in the sense of* dim*, mapping onto an* n*-dimensional metrizable space with a countable base; e) every* n*-dimensional factor-space* G/H *of weight* τ *of a locally bicompact group* G *by a closed subgroup* H *is representable as the sum of pairwise disjoint open-and-closed subsets in* G/H*, each of which is homeomorphically embeddable in* $P^{n\tau}$*.****
3) The bicompactum $P^{n\tau}$ is homogeneous (i.e. some group of transformations acts transitively on it).
4) $P^{n\tau}$ is a linearly (but not locally) connected dyadic bicompactum, mapped zero-dimensionally onto the n-dimensional torus C^n.
5) $P^{n\tau}$ is an $(n+1)$-fold image of a zero-dimensional bicompactum.*****

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* A space P is called universal for a certain class of spaces if every space of this class is homeomorphic to some subset of the space P.

** We note that for such bicompacta all dimensions coincide (see Theorem 7).

*** For n-dimensional, in the sense of dim, metrizable spaces of weight τ, a universal space was constructed earlier in (1).

**** See also Theorems 8, 10.

***** Consequently, all subsets of the space $P^{n\tau}$ are $(n+1)$-fold closed images of zero-dimensional, in the sense of ind, completely regular spaces. Hence it follows, for example: a) all n-dimensional bicompacta that are mapped zero-dimensionally onto compacta (including all $P^{n\tau}$) are perfectly n-dimensional in the sense of P. S. Aleksandrov and V. I. Ponomarev (2); b) an n-dimensional factor-space G/H of a locally bicompact group G by a closed subgroup H is an $(n+1)$-fold closed image of a zero-dimensional, in the sense of dim, space decomposing into the sum of pairwise disjoint open-closed final-compact sets. The latter assertion was established somewhat earlier by E. Sklyarenko.

6) $P^{n\tau}$ is the limit of the spectrum $\Sigma = \{P_\alpha, \omega_\alpha^\beta\}$ of n-dimensional polyhedra P_α, taken in certain triangulations, whose projections ω_α^β are simplicial with respect to certain triangulations of the polyhedra P_β and P_α. Moreover, the projections ω_α^β are nondegenerate (i.e., the images of the simplexes of the polyhedron P_β have the same dimension as their inverse images) and are "onto" mappings.

Theorem 2. Among all bicompacta of weight $\leq \tau$, zero-dimensionally mapping into an arbitrary fixed bicompactum Y_0 of weight $\leq \tau$, there exists a universal bicompactum Y_1 with $\operatorname{ind} Y_1 = \operatorname{ind} Y_0$, $\dim Y_1 = \dim Y_0$. If the bicompactum Y_0 is perfectly normal, then $\operatorname{ind} Y_1 = \operatorname{Ind} Y_1$. In particular, for all bicompacta of weight $\leq \tau$ that map zero-dimensionally onto compacta, and also for all metric spaces of weight $\leq \tau$, there exists a universal bicompactum P^τ, mapping zero-dimensionally onto the Hilbert cube I^∞*.

It is known that an open-and-closed mapping cannot raise the dimension of a space that is zero-dimensional in the sense of ind (or dim). There exist only two examples ([3],[4]) of open zero-dimensional mappings that raise dimension, and in both ([3]) and ([4]) a one-dimensional continuum maps zero-dimensionally and openly onto a two-dimensional one (in ([4]), onto the square), i.e., the dimension is raised by one.

Theorem 3. 1. Every n-dimensional compactum, $n = 1, 2, ...$, is an open and zero-dimensional image of some one-dimensional compactum.

2. There exists an infinite-dimensional compactum that is an open and zero-dimensional image of a one-dimensional compactum.
3. Every n-dimensional metric space with a countable base, $n = 1, 2, ...$, is a zero-dimensional, compact, open and closed image of some one-dimensional metric space with a countable base.
4. Each bicompactum $P^{n\tau}$, $n = 1, 2, ...$, is an open and zero-dimensional image:

$$P^{n\tau} = f(X_{pn\tau})$$

of some bicompactum $X_{pn\tau}$ of weight τ with

$$\dim X_{pn\tau} = \operatorname{ind} X_{pn\tau} = \operatorname{Ind} X_{pn\tau},$$

and, for all $y \in P^{n\tau}$, the sets $f^{-1}(y)$ are metrizable, and $X_{pn\tau}$ maps zero-dimensionally onto a compactum.

5. From item 3 it follows that any subset A of the bicompactum $P^{n\tau}$ is a zero-dimensional, bicompact, open and closed image:

$$A = f(X_A)$$

of some space X_A of weight τ, one-dimensional in the sense of ind, and, for all $y \in A$, the sets $f^{-1}(y) \subseteq X_A$ are metrizable and X_A maps by a decomposing mapping (see Definition 1) onto a one-dimensional metric space with a countable base.

In particular:

6. Every n-dimensional bicompactum Y of weight τ, mapping zero-dimensionally onto a compactum, is an open and zero-dimensional image:

$$Y = f(X_Y)$$

of some bicompactum X_Y of weight τ, one-dimensional in any sense, and, for all $y \in Y$, the sets $f^{-1}(y) \subseteq X_Y$ are metrizable and X_Y possesses a zero-dimensional mapping onto a compactum.

7. Any n-dimensional in the sense of dim metric space R of weight τ is a zero-dimensional, bicompact, open and closed image:

$$R = f(X_R)$$

of some metric space X_R of weight τ, one-dimensional in the sense of ind, and X_R possesses a decomposing mapping onto a one-dimensional metric space with a countable base.

Theorem 4. If a mapping f of a normal space X onto a space Y is closed and zero-dimensional in the sense of ind, and if bY is some bicompact Hausdorff extension of the space Y, then there exists a bicompact extension bX of the space X such that the mapping f can be extended to a zero-dimensional mapping of the bicompactum bX onto the bicom- the compactum bY. The extension bX may be assumed to be perfect (^{5}). If $\dim f = 0$, then one may assume

$$w(bX) \le \max(w(bY)^*, w(X))^{**}.$$

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* See also Theorems 9 and 10.

Theorem 5. In order that a mapping $f : X \to Y$ of a bicompactum X be zero-dimensional, it is necessary and sufficient that, for every open covering ω of the space X, the mapping f be representable as the superposition of mappings $g : X \to Z$ and $h : Z \to Y$, where g is an ω-mapping, and the mapping h is finite-to-one***.

Theorem 6. The following assertions are equivalent:

1. The bicompactum X has $\dim X \leq n$ and there exists a zero-dimensional mapping of the bicompactum X onto a compactum.
2. The bicompactum X is the limit of a spectrum

$$S = \{\Phi_\alpha,\ \delta_\alpha^\beta\}$$

of n-dimensional compacta Φ_α with zero-dimensional projections δ_α^β.

3. The bicompactum X is the limit of a spectrum

$$S' = \{\Phi'_\alpha, \delta'^\beta_\alpha\}, \quad \alpha \in \mathfrak{A},$$

of n-dimensional compacta Φ'_α, with finite-to-one and piecewise topological**** projections δ'^β_α, which are mappings "onto," and moreover each projection δ'^β_α is representable as the superposition of a finite number of twofold projections

$$\delta^\alpha_{\alpha_1},\ \delta^{\alpha_1}_{\alpha_2}, \ldots, \delta^\beta_\alpha.$$

One may assume that in the set $\mathfrak{A}$ there is a unique minimal element α_0 and that the compactum Φ'_{α_0} is a subset of the n-dimensional cube I^n.

4. The bicompactum X is the limit of a spectrum

$$\Sigma = \{P_\alpha, \pi_\alpha^\beta\}, \quad \alpha \in \mathfrak{A},$$

of n-dimensional polyhedra P_α, given in certain triangulations, and the projections π^β_α of the spectrum Σ are non-degenerate and simplicial (with respect to certain subdivisions of the polyhedra P_β and P_α) mappings. One may assume that in the set $\mathfrak{A}$ there is a unique minimal element α_0 and that the polyhedron P_{α_0} lies in the cube I^n.

In particular, if the bicompactum X is a compactum, then in items 3 and 4 the spectra S' and Σ may be assumed countable and ordered.

Definition 1. A mapping $f : X \to Y$ is called **splitting** (^{6}) if for every point $x \in X$ and every neighborhood Ox of it there exists a neighborhood Oy of the point $y = f(x)$ such that

$$f^{-1}(Oy) = O' \cup O'', \quad O' \cap O'' = \Lambda, \quad x \in O' \subseteq Ox,$$

and the sets O' and O'' are open in X.

Particular cases of splitting mappings are, for example, closed zero-dimensional mappings in the sense of ind of normal spaces. Note that if a mapping $f : X \to Y$

is splitting, then the space X is necessarily (completely) regular, provided the space Y is such.

Theorem 7. If the space X admits a splitting mapping onto a metric space Y, and for every closed subset $F \subseteq X$ the relation $\dim F \leq \operatorname{ind} F$ holds, then

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X$$

(and even $\dim F = \operatorname{ind} F = \operatorname{Ind} F$).

In particular, if a bicompact, or finally compact, or strongly paracompact, or, finally, completely paracompact (7) Hausdorff space X admits a splitting mapping onto a metric space Y, then

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X$$

*****.

Definition 2. Let the mapping $f : X \to Y$ be splitting. A system

$$\{\,{}_{\alpha}O_0,\ {}_{\alpha}O',\ {}_{\alpha}O''\,\}, \quad \alpha \in \mathfrak{A},$$

of open sets ${}_{\alpha}O_0 \subseteq Y$, ${}_{\alpha}O', {}_{\alpha}O'' \subseteq X$ such that

$$f^{-1}({}_{\alpha}O_0) = {}_{\alpha}O' \cup {}_{\alpha}O'', \quad {}_{\alpha}O' \cap {}_{\alpha}O'' = \Lambda,$$

will be called a **base of the splitting** f, if for every point $x \in X$ and every neighborhood Ox of it there will be an index α such that $x \in {}_{\alpha}O' \subseteq Ox$. The least cardinality of all possible bases of the mapping f will be called the **resolving weight** $cw(f)$ of the mapping f*.

Theorem 8. $P^{n\tau}$ is a universal space for all completely regular spaces possessing a resolving mapping f with $cw(f) \leq \tau$ onto an n-dimensional space with a countable base.

Theorem 9. For all completely regular spaces (in particular, bicompacts of weight τ) possessing a resolving mapping f with $cw(f) \leq \tau$** onto a weakly

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* $w(X)$ denotes the weight of the space X.

** Theorem 4 is generalized in Theorem 11.

*** For compacta this assertion was proved by A. Chernavskii.

**** A mapping $f : X \to Y$ is called piecewise topological if the space X is representable as the sum of a finite number of its closed subspaces, on each of which the mapping f is a homeomorphism.

***** For the case where the space X is completely paracompact, the theorem was proved by A. Zarelua.

countable-dimensional metric space with a countable base, there exists a universal space of weight τ ($\tau \geq \mathfrak{c}$).

Theorem 10. Suppose there is a space X_0 and a system $\{X_\theta\}$, $\theta \in \Theta$, of spaces X_θ possessing resolving mappings f_θ with $cw(f_\theta) \leq \tau$ into the space X_0. Then there exists a space P, universal for all spaces X_θ, and $w(P) \leq \max(w(X_0), \tau)$. If X_0 is a normal space, then P may be taken to be a bicompact with $\dim P \leq \dim X_0$.

Theorem 11. If a mapping f of a space X onto a space Y is resolving and bY is some bicompact (not necessarily Hausdorff) extension of the space Y, then there exists a bicompact extension bX of the space X to which the mapping f extends in such a way that the extended mapping is closed, bicompact, and zero-dimensional in the sense of ind***. Moreover, $w(bX) \leq \max(w(bY), cw(f))$ and $\operatorname{ind} bX \leq \operatorname{ind} bY$. If the space Y is completely regular and bY is a bicompact, then the bicompact bX may be taken to be a perfect extension (^{5}) of the space X.

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REFERENCES CITED

[1] J. Nagata, J. f. reine u. angew. Math., **204**, H. 1/4 (1960).
[2] P. S. Aleksandrov, V. I. Ponomarev, Sibirsk. matem. zhurn., **1**, No. 3, 3 (1960).
[3] A. Kolmogoroff, Ann. Math., **38**, No. 1, 36 (1937).
[4] L. V. Keldysh, Izv. AN SSSR, ser. matem., **23**, 165 (1959).
[5] E. Sklyarenko, DAN, **137**, No. 1, 39 (1961).
[6] A. Zarelua, DAN, **144**, No. 4 (1962).
[7] A. Zarelua, DAN, **141**, No. 4, 777 (1961).

* Note that $cw(f) \leq \max(w(Y), \operatorname{power} X)$.

** For bicompacts of weight τ, it is always the case that $cw(f) \leq \tau$.

*** This part of the theorem was proved by the author for closed zero-dimensional mappings of normal spaces (see Theorem 4) and then was carried over by A. Zarelua to resolving mappings of completely regular spaces, and by the author—even to resolving mappings of arbitrary T_1-spaces.

Note: Figure translations are in progress. See original paper for figures.

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