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Abstract

Full Text

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L. P. Lebedev

An Effective Method for Constructing Best Methods of Approximation

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By virtue of D. Jackson's theorems on the influence of the structural properties of a function $f(x)$ on the order of its approximation by trigonometric polynomials, we have

$$\mathcal{E}U_n(KW^{(r)}H^{(\alpha)}, x) = \sup_{f \in KW^{(r)}H^{(\alpha)}} \left| f(x) - U_n(f; x) \right| = O\left(\frac{1}{n^{r+\alpha}}\right), \qquad (1)$$

where $KW^{(r)}H^{(\alpha)}$ $(r = 0, 1, 2, \dots,\ 0 \le \alpha < 1)$ is the class of functions $f(x)$ of period 2π having a derivative of order r satisfying a Lipschitz condition of degree α with constant K, and $U_n(f; x)$ is a polynomial method of best approximation of order not exceeding n.

Of interest is the problem of actually constructing a sequence of trigonometric polynomials $U_n(f; x)$ realizing relation (1).

All concrete methods of summation of Fourier series known up to now of the form

$$U_n(f; x; \lambda) = \frac{a_0}{2}\lambda_0^{(n)} + \sum_{k=1}^{n} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx), \qquad (2)$$

where a_k and b_k are the Fourier coefficients of the function $f(x) \in C_{2\pi}$, for example the Fejér sums

$$\left(\lambda_k^{(n)} = 1 - \frac{k}{n}\right),$$

the de la Vallée Poussin sums

$$\left(\lambda_k^{(n)} = \frac{(n!)^2}{(n-k)!(n+k)!}\right),$$

the Bernstein–Rogozinskii sums

$$\left(\lambda_k^{(n)} = \cos\frac{k\pi}{2n+1}\right),$$

their numerous generalizations (including Zygmund' s method:

$$\lambda_k^{(n)} = 1 - \left(\frac{k}{n+1}\right)^p, \quad p = 1, 2, \dots$$

), as well as all linear positive operators, are not a solution of this problem. Thus, the order of approximation of functions $f(x) \in C_{2\pi}$ by the Fejér and de la Vallée Poussin operators cannot be better than $1/n$, by the Bernstein-Rogozinskii polynomials and by all linear positive operators not better than $1/n^2$, and by the Zygmund sums not better than $1/n^p$; that is, all these modifications of Fourier sums have the common defect that the order of approximation effected by them on the classes $KW^{(r)}H^{(\alpha)}$ is worse than the best one the better the function is. Let us note that approximation of functions of the classes $KW^{(r)}H^{(\alpha)}$ by the Fourier sums themselves does not have such a defect, since it has, as is known, order $\ln n/n^{r+\alpha}$.

This fact is apparently connected with the circumstance that whereas for all the methods mentioned

$$\lim_{n\to\infty} (1 - \lambda_k^{(n)}) n^m \neq 0, \quad m \geq m_0,$$

for the partial sums of the Fourier series we have

$$\lim_{n\to\infty} (1 - \lambda_k^{(n)}) n^m = 0, \quad m = 1, 2, \dots, \quad k = 0, 1, 2, \dots .$$

Thus, methods of summation of Fourier series of type (2) have hitherto been constructed in such a way that the removal of the defect of the Fourier partial sums (the absence of their uniform convergence on the whole class of continuous 2π-periodic functions) was carried out simultaneously with the loss of a valuable quality of this approximation apparatus—its good approximative properties on classes of differentiable functions.

In the present note a simple method is described for constructing polynomials of type (2), uniformly convergent on the whole class of continuous 2π-periodic functions, and realizing relation (1). In the course of the discussion this method appears as a special case of more general considerations in approximation theory.

Definition. A class of methods of summation of Fourier series of type (2) will be called the class $e_n(f; x; \lambda)$, if

$$\lambda_k = \lambda_k^{(n)} = 1 - e^{\varphi(n,k)}, \qquad k = 1, 2, \dots, n; \qquad \lambda_0^{(n)} = 1, \quad \lambda_{n+1}^{(n)} = 0 \quad (3)$$

and on $[1, n]$ $\varphi(n, x)$ is twice differentiable, $\varphi'(n, x) \geq 0$, and $\varphi''(n, x)$ is continuous.

It is easy to see that all classical methods (if, instead of the Valle-Poussin factors, one takes a construction close to them with differentiable function $\lambda_k^{(n)} = e^{-k^2/n}$)

belong to the class $e_n(f;x;\lambda)$. However, of greatest interest are the summation methods of the class $e_n(f;x;\lambda)$ proper, i.e., those for which the simplest representation of the convergence factors is the form (3).

Theorem 1. For the relation

$$e_n(f;x;\lambda) \xrightarrow[n\to\infty]{} f(x) \in C_{2\pi} \tag{4}$$

to hold uniformly with respect to real x, it is necessary that the conditions

$$\lim_{n\to\infty} \varphi(n,k) = -\infty, \qquad k = 1,2,\dots; \tag{5}$$

$$\lambda_{n-m}^{(n)} \ln \frac{n}{m+1} = O(1), \qquad \frac{m}{n} \xrightarrow[n\to\infty]{} 0 \tag{6}$$

be satisfied.

The proof of the necessity of condition (6) follows directly from the equality

$$\sum_{k=1}^{n} \frac{\lambda_k^{(n)}}{n-k+1} = \lambda_n^{(n)} \ln n + \sum_{k=1}^{n-1} \Delta_k \ln \frac{n}{n-k} + O(1),$$

$$\Delta_k = \lambda_k^{(n)} - \lambda_{k+1}^{(n)}.$$

The following theorem is valid for any method (2) with a function $\lambda_x^{(n)}$ twice differentiable on $[1,n]$ (λ_x'' is continuous).

Theorem 2. Suppose it is known that the function λ_x'' can have only a finite number of real zeros in the interval $[1,n]$. Then, for relation (4) to hold, it is sufficient that conditions (5), (6) be satisfied with $m=0$, and

$$x\lambda_x' = O(1), \qquad x \le n. \tag{7}$$

If λ_x'' does not change sign in the interval $[1,n]$ (the methods of Fejér, Zygmund, Bernstein–Rogosinski), then it is sufficient to require the fulfil-

condition (7) only for $x=n$. If the roots x_i ($i=1,2,\dots,i_0$) of the second derivative λ_x'' are known, then (7) is checked only for $x=x_i$ and $x=n$; if, moreover, $\lambda_x' \leqslant 0$ on $[1,n]$ and λ_x'' is nonnegative on $[x_{i_0},n]$ (the Vallée-Poussin method in the interpretation given above), then requirement (7) need be satisfied only for $x=x_i$.

Let us add also that, if $\Delta_k^2 \geqslant 0$, $k=1,2,\dots,n-2$, and $\lambda_x' \leqslant 0$ ($\varphi'(n,x) \geqslant 0$) on $[1,n]$, then for the fulfillment of relation (4) it is necessary and sufficient that conditions (5) and (6) be observed for $m=0$.

The proof of all these recommendations is easily obtained from the relations

$$\int_0^{\pi}\left|\frac{1}{2}+\sum_{k=1}^{n}\lambda_k^{(n)}\cos kt\right|dt\leqslant\frac{2}{\pi}\left|\lambda_n^{(n)}\right|\ln n+\frac{\pi}{2}\sum_{k=0}^{n-2}(k+1)\left|\Delta_k^2\right|+\frac{\pi}{2}n\left|\Delta_{n-1}\right|,$$

$$\Delta_k^2=\Delta_k-\Delta_{k+1}=\gamma\lambda_{k+\theta}'',\qquad k=1,2,\dots,n-2,\qquad 0<\gamma<2,\qquad 0<\theta<2.$$

Theorem 3. Suppose that for the method $e_n(f;x;\lambda)$ the following conditions are satisfied:

$$\lambda_n^{(n)}\ln n=O(1),\tag{8}$$

$$e^{\varphi(n,k)}=O\left[\left(\frac{k}{n_0}\right)^m\right],\tag{9}$$

or, in an equivalent form,

$$\varphi(n,k)>O\left(\ln\frac{n_0}{k}\right),\qquad\varphi'(n,k)\leqslant O\left[\left(\frac{n_0}{k}\right)^{p_0}\frac{1}{k^{t_0}}\right],\tag{10}$$

where $n_0\leqslant O(n)$; $k<O(n_0)$; m is any positive number; p_0 and $t_0\geqslant 1$ are certain fixed real numbers. (It is easy to see that $e^{\varphi(n,k)}=O(1)$, if $n\geqslant k\geqslant O(n_0)$.)

Then, for any integer $r\geqslant 0$ and $0\leqslant\alpha<1$, the inequality

$$\mathcal{E}_n=\begin{cases}\mathcal{E}_{e_n}\left(KW^{(r)}H^{(\alpha)},x\right)\\ \mathcal{E}_{e_n}\left(\widetilde{KW}^{(r)}H^{(\alpha)},x\right)\end{cases}\leqslant O\left(\frac{n^2}{n_0^{r+\alpha+2}}\right).\tag{11}$$

is valid.

The proof consists in estimating the sums entering into the formulation of A. F. Timan' s theorem (([1]), § 8.5.1); here one uses the equality

$$\Delta^2\mu_k^{(n)}=\bar{\gamma}f''(k+\bar{\theta}),\qquad 0<\bar{\gamma}<2,\qquad 0<\bar{\theta}<2,\qquad k=1,2,\dots,n-2;$$

$$f(x)=\frac{e^{\varphi(n,x)}}{x^r}.$$

Estimate (11) is the more accurate, the better the method $e_n(f;x;\lambda)$.

Corollary. If in the conditions of Theorem 3 $n_0 = O(n)$, then the method $e_n(f; x; \lambda)$, on the classes $KW^{(r)}H^{(\alpha)}$ and the conjugate classes, carries out uniform approximation of the best possible order, i.e. realizes relation (1).

Examples. Everywhere $\lambda_0^{(n)} = 1$, $\lambda_{n+1}^{(n)} = 0$, and the formulas given below are valid for $k = 1, 2, \dots, n$.

1. $\lambda_k^{(1)} = 1 - e^{-e^{n-k}/n}$. Condition (6) is violated.
2. $\lambda_k^{(2)} = 1 - e^{-e^{n-2k}}$. The conditions of Theorem 2 are violated.
3. $\lambda_k^{(3)} = 1 - e^{-n/k^\gamma}$, $\gamma > 1$. The conditions of Theorems 2 and 3 are satisfied for
$$n_0 = O(n^{1/\gamma}); \qquad \mathcal{E}_n \ll O\left(\frac{n^2}{n^{(r+\alpha+2)/\gamma}}\right).$$
4. $\lambda_k^{(4)} = 1 - e^{-np/\ln n \cdot k^p}$, $p > 0$. The conditions of Theorems 2 and 3 are satisfied for
$$n_0 = O\left[\frac{n}{(\ln n)^{1/p}}\right]; \qquad \mathcal{E}_n \ll O\left[\frac{(\ln n)^{(r+\alpha+2)/p}}{n^{r+\alpha}}\right].$$
5. $\lambda_k^{(5)} = 1 - e^{-p \ln^\gamma (n/k)}$, $p > 0$, $\gamma > 0$—a generalization of the methods of Fejér ($p = 1$, $\gamma = 1$) and Zygmund ($\gamma = 1$). The conditions of Theorem 2 are satisfied and, if $\gamma > 1$, those of Theorem 3 for $n_0 = O(n)$; for $\gamma > 1$
$$\mathcal{E}_n = O\left(\frac{1}{n^{r+\alpha}}\right).$$

The last equality is also valid for the method:

$$6. \quad \lambda_k^{(6)} = 1 - e^{-n/k + k/n}.$$

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REFERENCES

1. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960.

Note: Figure translations are in progress. See original paper for figures.