TWO-DIMENSIONAL FORMAL ABELIAN GROUPS

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Abstract

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MATHEMATICS

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TWO-DIMENSIONAL FORMAL ABELIAN GROUPS

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1. An n-dimensional formal group G over a field k is [3] a pair $(\mathfrak{o}, T)$ consisting of the ring $\mathfrak{o}$ of formal power series in n variables with coefficients in the field k and a k-homomorphism $T : \mathfrak{o} \to \mathfrak{o}\widehat{\otimes}\mathfrak{o}$ of this ring into its completed tensor product with itself. The mapping T must satisfy certain axioms, which together mean that the k-linear space $\mathfrak{o}$ is a k-coalgebra with composition law T. The class of formal abelian groups over the field k forms a category. Morphisms $\varphi : G_1 \to G_2$ of this category, where $G_1 = (\mathfrak{o}_1, T_1)$, $G_2 = (\mathfrak{o}_2, T_2)$, are in one-to-one correspondence with k-homomorphisms $\varphi : \mathfrak{o}_2 \to \mathfrak{o}_1$ that commute with the structure mappings:

$$(\varphi\widehat{\otimes}\varphi) \circ T_2 = T_1 \circ \varphi.$$

Dieudonné [1] showed that the image $\varphi(\mathfrak{o}_2)$ is always a ring of formal power series over k. If

$$\dim \mathfrak{o}_1 = \dim \mathfrak{o}_2 = \dim \varphi(\mathfrak{o}_2),$$

the morphism φ is called an isogeny. We shall assume that the field k is algebraically closed. If its characteristic is zero, then every formal abelian group over it is isomorphic to a direct sum of additive groups (a formal group is called abelian if the coalgebra $\mathfrak{o}$ is abelian). If, however, the characteristic p of the field k is different from zero, the classification is much more complicated.

We shall call two formal groups G_1, G_2 **isogenous** if there exists a third group G_3 and two isogenies $\varphi_1 : G_3 \to G_1$ and $\varphi_2 : G_3 \to G_2$. Isogeny is an equivalence relation. Dieudonné [2] showed that every abelian formal group is isogenous to a direct sum of indecomposable groups. Every indecomposable group of dimension n is isogenous to one of the groups $G_{n,m}$, where m runs through all nonnegative integer values relatively prime to n, and the symbol ∞.

In what follows, k denotes an algebraically closed field of characteristic $p > 0$, and all formal groups and their morphisms are assumed to be defined over this field.

The principal unsolved question in the theory of formal abelian groups is their classification up to isomorphism. Isogenous one-dimensional groups are isomorphic. The purpose of this note is to describe all two-dimensional formal abelian groups. The method of classification relies essentially on Dieudonné' s fundamental fact, established by him, that the category of formal abelian groups is equivalent to the category of modules over a certain complete topological ring. Most of the results set out below generalize to the case of groups of arbitrary dimension. We restrict ourselves below to the case of dimension two both because, in the general case, some technical difficulties have not yet been overcome, and because for two-dimensional groups it is possible to trace many details of the picture, which becomes sharply more complicated for larger values of the dimension.

2. If a two-dimensional abelian group G is isogenous to a direct sum of one-dimensional groups, and if one of the direct summands is the group $G_{1,0}$, then the group G is isomorphic to this direct sum. Thus it suffices to classify the groups isogenous to one of the groups of the form

$$G_{2,m} \quad (1 \le m < \infty,\ m \text{ odd}); \qquad G_{2,\infty}; \qquad G_{1,m_1}+G_{1,m_2} \quad (m_1 \ne m_2;\ 1 \le m_1 m_2 \le \infty); \qquad G_{1,m}+G_{1,m} \quad (1$$

We shall call these groups basic. If the group G' is isogenous to one—

one of the basic groups G, then there exists an isogeny $\varphi : G' \to G$. Let $G' = (\mathfrak{o}', T')$, $G = (\mathfrak{o}, T)$; $\mathfrak{o} \subset \mathfrak{o}'$ is the embedding induced by the isogeny φ. The **height** of the isogeny φ is the least integer h such that $\mathfrak{o}' p^h \subset \mathfrak{o}$. Such a number always exists. Let us introduce the notion of a **primitive isogeny**. In the case of an indecomposable basic group G, an isogeny $\varphi : G' \to G$ is called primitive if $\mathfrak{o} p^{-1} \not\subset \mathfrak{o}'$. If the basic group G is decomposable, then its ring $\mathfrak{o}$ is represented in the form of a completed tensor product $\mathfrak{o}_1 \widehat{\otimes} \mathfrak{o}_2$ of rings of one-dimensional groups—this representation is ambiguous only for the basic group $G_{1,m} + G_{1,m}$. In this case we shall call an isogeny $\varphi : G' \to G$ primitive if $\mathfrak{o}_1 p^{-1} \otimes 1 \not\subset \mathfrak{o}'$ and $1 \otimes \mathfrak{o}_2 p^{-1} \not\subset \mathfrak{o}'$ (for any decomposition).

Lemma 1. *Let the group G' be isogenous to the basic group G. Then there exists a primitive isogeny $\varphi : G' \to G$.*

The first stage of the classification consists in describing the primitive isogenies.

Lemma 2. a) *In the case $G = G_{2,m}$, the height of a primitive isogeny does not exceed m; in the case $G = G_{1,m_1} + G_{1,m_2}$, the height of a primitive isogeny does not exceed* $\min(m_1, m_2)$.

- b) *The set of classes of equivalent primitive isogenies $G' \to G$ of a given height h can be endowed with the structure of an affine space over the field k. The dimension of this space is $[h/2]$, if $G = G_{2,m}$, and h, if $G = G_{1,m_1} + G_{1,m_2}$. In the latter case one must remove from this space the hypersurface whose points correspond to nonprimitive isogenies.*

Remark 1. We shall call two isogenies $\varphi : G' \to G$ and $\psi : G'' \to G$ equivalent if there exists an isomorphism $\varepsilon : G' \to G''$ such that $\varphi = \psi \circ \varepsilon$.

Remark 2. In the course of the proof of Lemma 2, a certain system of coordinates is chosen in the affine spaces described. In what follows, when speaking of the coordinates of a primitive isogeny $G' \to G$ of height h, we shall always mean the coordinates of the corresponding point $(a_1, \dots, a_d)$ in the affine space, $d = [h/2]$ or h.

For a given group G', generally speaking, there exist several primitive isogenies $G' \to G$. We shall call such an isogeny **minimal** if it has the least possible height.

Lemma 3. a) *Let $G = G_{2,m}$ $(1 \leq m < \infty,\ m$ odd). The index of a primitive isogeny $\varphi : G' \to G$ with coordinates $(a_1, \dots, a_{[h/2]})$ is the integer j, $0 \leq j \leq [h/2]$, for which $a_j \neq 0,\ a_{j+1} = \dots = a_{[h/2]} = 0$ ($j = 0$, if $a_1 = \dots = a_{[h/2]} = 0$). A primitive isogeny of height h and index j is minimal if and only if $h \leq \frac{m-1}{2} + j$ and $a_j^{p^{m+2}} \neq a_j$ for $j \neq 0$. The index of a minimal isogeny $\varphi : G' \to G$ is an invariant of the group G'.*

b) *Let $G = G_{2,\infty}$. An isogeny $\varphi : G' \to G$ is minimal if and only if all its coordinates are zero.*

c) *Let $G = G_{1,m_1} + G_{1,m_2}$. An isogeny $\varphi : G' \to G$ of height h is minimal if and only if $a_h^{p^{m+1}} \neq a_h$.*

It remains now to find out which minimal isogenies correspond to isomorphic formal groups. We shall identify a minimal isogeny with the corresponding point in the affine space. Denote by the symbol $A(m; h, j)$ the algebraic set of minimal isogenies $G' \to G_{2,m}$ of height h and index j, and by the symbol $A(m_1, m_2; h)$ the set of minimal isogenies $G' \to G_{1,m_1} + G_{1,m_2}$ of height h.

Theorem. a) *Let $G = G_{2,m}$, $1 \leq m < \infty$. For every pair of integers h, j satisfying the inequalities $0 \leq 2j \leq h \leq \frac{m-1}{2} + j$, $1 \leq h \leq m$, there exists a finite group of biregular automorphisms $\Gamma(m; h, j)$ of the set $A(m; h, j)$ such that isogenies $\varphi \in A(m; h, j)$ and $\psi \in A(m; h', j')$ correspond to isomorphic groups if and only if*

case, if $h = h'$, $j = j'$, and φ, ψ are transformed into one another by operations of the group $\Gamma(m; h, j)$.

Thus the "moduli space" of groups isogenous to the group $G_{2,m}$ is a (finite-dimensional) algebraic variety over the field k:

$$\bigcup A(m, h, j)/\Gamma(m; h, j).$$

(Obviously,

$$\dim A(m; h, j)/\Gamma(m; h, j) = \dim A(m; h, j) = j.)$$

b). Let $G = G_{2,\infty}$. Then distinct minimal isogenies correspond to nonisomorphic formal groups. Thus the set of isomorphism classes of groups isogenous to the group $G_{2,\infty}$ is in one-to-one correspondence with the set of nonnegative integers h. (The corresponding composition law has the form

$$Tx_1 = x_1 \otimes 1 + 1 \otimes x_1; \qquad Tx_2 = x_2 \otimes 1 + 1 \otimes x_2 + F^{p^h}(x_1 \otimes 1, 1 \otimes x_1),$$

where
$$F(x,y) = \frac{x^p + y^p - (x+y)^p}{p} \pmod{p}.)$$

c). Let $G = G_{1,m_1} + G_{1,m_2}$. For any integer $h \leq \min(m_1, m_2)$ there exists a finite group $\Gamma(m_1, m_2; h)$ of biregular automorphisms of the set $A(m_1, m_2; h)$ such that the isogenies $\varphi \in A(m_1, m_2; h)$ and $\psi \in A(m_1, m_2; h')$ correspond to isomorphic formal groups if and only if $h' = h$, and φ, ψ are transformed into one another by operations of the group $\Gamma(m_1, m_2; h)$. Thus the moduli space of groups isogenous to the group $G_{1,m_1} + G_{1,m_2}$ is the algebraic variety
$$\bigcup A(m_1, m_2; h)/\Gamma(m_1, m_2; h).$$

(Obviously,
$$\dim A(m_1, m_2; h)/\Gamma(m_1, m_2; h) = \dim A(m_1, m_2; h) = h.)$$

Remark 1. The groups $\Gamma(m; h, j)$ and $\Gamma(m_1, m_2; h)$, and their action on the corresponding spaces, can be described explicitly. However, this would require entering into the details of the proof.

Remark 2. It would be interesting to find out whether the space
$$\bigcup A(m; h, j)/\Gamma(m; h, j)$$
is a scheme, defining its intrinsic topology by means of the concept of the moduli space due to Grothendieck.

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Note: Figure translations are in progress. See original paper for figures.