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1962

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

HYDROMECHANICS

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THE FORMATION OF A SHOCK WAVE UPON REFLECTION OF A WEAK DISCONTINUITY FROM A SONIC LINE

(Presented by Academician L. D. Landau, December 15, 1961)

In the work of L. D. Landau and E. M. Lifshitz (1) (see also (2)), the reflection of a weak discontinuity (a jump in the first derivatives of the velocity with respect to the coordinates) from a sonic line, i.e., from the line where the flow velocity is equal to the local speed of sound, was studied. In doing so, the authors restricted themselves to the case in which the jump of the derivatives in the direction of the flow on the incident discontinuity is positive; in this case the discontinuity is reflected from the sonic line in the form of a peculiar weaker singularity. A different picture obtains if the jump of the derivatives in the discontinuity incident on the sonic line is negative. Below it will be shown that such a discontinuity is reflected from the sonic line in the form of a shock wave, whose intensity near the point of reflection is exponentially small*.

Near the sonic line the flow is described by the Euler-Tricomi equation** $\Phi_{\eta\eta} - \eta\Phi_{\theta\theta} = 0$. We choose the origin of coordinates (x, y) at the point of intersection of the discontinuity with the sonic line; to this point there corresponds the origin of coordinates also in the hodograph plane (η, θ). In Fig. 1, the incident discontinuity in the physical plane corresponds in the hodograph plane to the characteristic (Oa), $\theta = \frac{2}{3}\eta^{3/2}$. The jump of the first derivatives of the velocity with respect to the coordinates is expressed in terms of the jump of the second derivatives of the potential Φ with respect to the variables θ, η. The solution of the Euler-Tricomi equation corresponding to such a singularity has, near the characteristic (Oa), the form (1):

Fig. 1

$$\Phi = -A\eta\theta - B_{(1,1')}\theta^{11/6}\xi^2 F\left({}^{13}/_{12},\ {}^{19}/_{12},\ 3;\ \zeta\right). \tag{1}$$

The coefficients B_1 and $B_{1'}$ characterize the solutions in regions 1 and $1'$ in Fig. 1 ($\xi = 1 - 4\eta^3/9\theta^2$).

The behavior of the solution near the characteristic (Ob), $\theta = -\frac{2}{3}\eta^{3/2}$, is determined by analytic continuation of the solutions (1) into regions 2 and $2'$. Neglecting terms beginning with third order in ζ, we obtain near (Ob)

$$\Phi = -A\theta\eta + \frac{C_{(2,2')}}{\pi}(-\theta)^{11/6}\left\{\xi^2 \ln|\xi| - \frac{2^9 \cdot 34}{385} + \right.$$

$$\left. + \frac{288}{7}\xi + \xi^2\left[2\gamma - 2\ln 2 + \frac{3573}{770} + \psi\left(\frac{17}{6}\right) + \psi\left(\frac{13}{6}\right)\right]\right\} =$$

$$= -A\theta\eta + \frac{C_{(2,2')}}{\pi}(-\theta)^{11/6}\left\{\xi^2 \ln|\xi| - \tau + \lambda\xi + \delta\xi^2\right\}. \qquad (2)$$

Here $C_2 = B_1$, $C_{2'} = B_{1'}/2$; the numerical coefficients are equal to: $\tau = 108$, $\lambda = 41.1$, $\delta = 5.89$.

* The possibility of the formation of a shock wave upon reflection of singularities from a sonic line was noted by Guderley (^{3}).
** In what follows we use the notation of (^{1},^{2}).

The condition of positivity of the Jacobian $\Delta = \partial(x,y)/\partial(\eta,\theta)$ is automatically satisfied near (Oa), where $\Delta \simeq A^2$. Near (Ob), Δ has the form

$$\Delta \simeq A^2 - AC\frac{16}{\pi}\left(\frac{3}{2}\right)^{1/3}(-\theta)^{1/6}\ln|\xi|.$$

In $(1,2)$ it is assumed that $AC > 0$. Then $\Delta > 0$ near (Ob). From the condition of continuity of the coordinates on the reflected characteristic it follows that $C_2 = C_{2'}$, and therefore $B_{1'} = 2B_1$. On the reflected weak discontinuity the derivatives of the velocity do not undergo a jump.

Suppose now that $AC < 0$. Choosing the positive direction of the x-axis in the direction of the gas velocity, we obtain from the condition that the incident singularity be incoming that $A > 0$.

For $C < 0$ the Jacobian vanishes on a limiting line exponentially close to the characteristic (Ob),

$$\left|\theta + \frac{2}{3}\eta^{3/2}\right| \sim |\theta| \exp\left[-\frac{A\pi(2/3)^{1/3}}{16|C||\theta|^{1/6}}\right]. \qquad (3)$$

The transition from region 2 to region $2'$ can be effected only in a shock wave. It is obvious that, in turn, the intensity of the shock wave is also exponentially small. Therefore, as before, the solution can be constructed only in the case when $B_{1'} = 2B_1 \equiv 2B$.

In Fig. 1 the region in the hodograph plane occupied by the shock wave is shaded. On the boundaries of this region four conditions must be satisfied, relating the parameters in the shock wave (^{2}): continuity of the coordinates x and y:

$$[\Phi_\theta] = 0, \qquad [\Phi_\eta] = 0; \tag{4}$$

continuity of the tangential components of the velocity at the front of the shock wave, equivalent to the condition of continuity of the potential $\varphi = -\Phi + \eta\Phi_\eta + \theta\Phi_\theta$:

$$[\varphi] = -[\Phi] + [\eta]\Phi_\eta + [\theta]\Phi_\theta = 0; \tag{5}$$

the equation of the shock polar

$$2(\theta_{2'} - \theta_2)^2 = (\eta_{2'} - \eta_2)^2(\eta_{2'} + \eta_2). \tag{6}$$

Near (Ob) it is convenient to pass to new variables:

$$v = \frac{2}{3}\eta^{3/2}, \qquad \xi_+ = \theta + \frac{2}{3}\eta^{3/2}. \tag{7}$$

In these variables the potential (2) takes the form:

$$\begin{gathered}\Phi = \left(\frac{3}{2}\right)^{2/3} A v^{2/3}(v - \xi_+) + \\ + \frac{B}{\pi}\left\{4\xi_+^2 v^{-1/6} \ln\frac{2|\xi_+|}{v} - 108 v^{11/6} + 116 v^{5/6}\xi_+ - 31,2 v^{-1/6}\xi_+^2\right\}.\end{gathered} \tag{8}$$

Expressions for the coordinates:

$$x = A(v - \xi_+) + \left(\frac{3}{2}\right)^{2/3} \frac{B}{\pi}\left\{-82,2 v^{7/6} + 43,1 v^{1/6}\xi_+ + 8\xi_+ v^{1/6} \ln\frac{|\xi_+|}{v}\right\}, \tag{9}$$

$$y = -\left(\frac{3}{2}\right)^{2/3} A v^{2/3} + \frac{B}{\pi}\left\{112 v^{5/6} + 62,4 v^{-1/6}\xi_+ + 8\xi_+ v^{-1/6} \ln\frac{|\xi_+|}{v}\right\}. \tag{10}$$

Substitute (8)–(10) into equations (4), (5) and neglect everywhere the terms corresponding to the third order in ξ_+ in the potential (8). Forming the combination

$$[\Phi_\eta] - v^{1/3}\left(\frac{3}{2}\right)^{1/3}[\Phi_\theta] = 0,$$

we find that

$$\delta v = -\frac{1}{2}\delta\xi_+, \tag{11}$$

where

$$\delta v = v_{2'} - v_2, \qquad \delta\xi_+ = \xi_{+2'} - \xi_{+2}.$$

Relation (11) satisfies the equation of the shock polar (6). In accordance with (3) we shall seek $\xi_{+2'}$ and ξ_{+2} in the form

$$\xi_{+2'} = a_{2'} v \exp\left[\frac{A\pi(2/3)^{1/3}}{16Bv^{1/6}}\right], \qquad \xi_{+2} = -a_2 v \exp\left[\frac{A\pi(2/3)^{1/3}}{16Bv^{1/6}}\right], \tag{12}$$

where $a_2, a_{2'} > 0$.

The combination $[\Phi_\eta] + v^{1/3}(3/2)^{1/3}[\Phi_\theta] = 0$, with (11) taken into account, yields the following equation relating a_2 and $a_{2'}$:

$$a_{2'}\ln a_{2'} + a_2 \ln a_2 - 0.59\,(a_2 + a_{2'}) = 0. \tag{13}$$

The second equation for finding a_2 and $a_{2'}$ is obtained from the condition

$$\frac{1}{4}[\zeta_+]\{[\Phi_\eta] + (3/2)^{1/3}v^{1/3}[\Phi_\theta]\} + [\eta]\Phi_{\eta 2} + [\theta]\Phi_{\theta 2} - [\Phi] = 0,$$

from which we obtain

$$a_2 a_{2'} \ln\frac{a_{2'}}{a_2} + \frac{1}{2}(a_{2'}^2 - a_2^2) = 0. \tag{14}$$

From (13) and (14) it is easy to determine the coefficients $a_2, a_{2'}$:

$$a_2 = a_{2'} = 1.84.$$

Let us now turn to the description of the flow in the physical plane. On the incident discontinuity we have (1) $x = -A\theta$, $y = -A\eta$. The equation of the discontinuity line is

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Fig. 2

Figure 2: Fig. 2

$$-y = (3/2)^{2/3} A^{1/3} (-x)^{2/3}.$$

Fig. 2

It is convenient to characterize the discontinuity of the velocity derivatives at the jump by means of the jump of the derivative $(\partial\eta/\partial x)_y$:

$$\left(\frac{\partial\eta}{\partial x}\right)_{y1} - \left(\frac{\partial\eta}{\partial x}\right)_{y1'} = 8.56\, BA^{-7/4}|y|^{-1/4}. \tag{15}$$

The difference from (1) consists in the fact that $B < 0$. Thus, depending on the sign of the jump of the velocity derivatives, the incident discontinuity is reflected either in the form of a shock wave ($B < 0$), or in the form of a weak logarithmic singularity, investigated in (1). To determine the form of the sonic line one may use the ready formulas (6) of (1). The equations of the two branches of the transition line are

$$y = -11.4\, BA^{-5/6}(-x)^{5/6}, \qquad y = 11.4\sqrt{3}\, BA^{-5/6}x^{5/6}.$$

The first branch corresponds in the hodograph plane to the half-axis $\eta = 0,\ \theta > 0$, the second to $\theta < 0,\ \eta = 0$.

The equation for the shock-wave front may be obtained from formulas (9) and (10). Discarding exponentially small terms and retaining only the first corrective power term, we find (cf. (4), (1))

$$-y = 1.31\, A^{1/3}x^{2/3} - 10.5\, BA^{-5/6}x^{5/6}.$$

The mutual arrangement of all the lines is shown in Fig. 2a. In Fig. 2b, for comparison, the flow pattern is given which was studied in (1) and which occurs if in (15) $B > 0$.

Finally, we give the formulas describing the change of the parameters at the shock wave:

$$\delta\eta = \eta_{2'} - \eta_2 = 1.57\left(\frac{A}{x}\right)^{1/3} \exp\left[0.224\,\frac{A^{7/6}}{Bx^{1/6}}\right],$$

$$\delta\theta = \theta_{2'} - \theta_2 = 0.92\,\frac{x}{A}\exp\left[0.224\,\frac{A^{7/6}}{Bx^{1/6}}\right].$$

In conclusion, the authors express their gratitude to Academician L. D. Landau for valuable advice and discussion of the work.

Received
6 XII 1961

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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