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**Abstract**

**Full Text**

**V. P. Chistyakov, N. P. Markova**

## **On Some Theorems for Inhomogeneous Branching Processes**

*(Presented by Academician A. N. Kolmogorov on 7 II 1962)*

A scheme of reproduction of particles is considered, in which each particle, independently of the others, during the time interval  $(t, t + \Delta t)$  is transformed into  $k$  particles ( $k$  natural and not equal to 1) with probability  $p_k(t)\Delta t + o(\Delta t)$  ( $\Delta t \rightarrow 0$ ), and undergoes no changes with probability

$$1 + p_1(t)\Delta t + o(\Delta t) \left( p_k(t) \geq 0, k \neq 1; p_1(t) \leq 0, \sum_{k=0}^{\infty} p_k(t) = 0 \right).$$

Such a process is a special case of branching processes.

Let  $\mu_{st}$  be the number of particles at time  $t$ , if at time  $s \leq t$  there was one particle, and let

$$F(s, t, x) = \sum_{n=0}^{\infty} P_n(s, t) x^n$$

be the generating function for  $\mu_{st}$ .  $F(s, t, x)$  satisfies the equation <sup>(1)</sup>

$$-\frac{\partial F(s, t, x)}{\partial s} = f(s, F(s, t, x)) \quad (1)$$

with initial condition

$$F(s, t, x)|_{s=t} = x \quad (0 \leq s \leq t < \infty).$$

Here

$$f(s, x) = \sum_{n=0}^{\infty} p_n(s) x^n.$$

The process is called degenerating if

$$\lim_{t \rightarrow \infty} P_0(s, t) = P_0(s) = 1.$$

This limit always exists. It is equal to 1 or different from 1 simultaneously for all  $s$ . Indeed, let  $s_0 < s$ . Then

$$P_0(s_0) = \sum_{k=0}^{\infty} P_k(s_0, s) P_0^k(s),$$

and, if  $P_0(s) = 1$ , then also  $P_0(s_0) = 1$ . At least one of the probabilities  $P_k(s_0, s)$  ( $k \geq 1$ ) is nonzero, since

$$\sum_{k=1}^{\infty} P_k(s_0, s) \geq \exp \left[ - \int_{s_0}^s p_0(v) dv \right] > 0.$$

Consequently,

$$\sum_{k=0}^{\infty} P_k(s_0, s) x^k$$

is strictly monotone in  $x$ , and therefore the equality

$$1 = P_0(s_0) = \sum_{k=0}^{\infty} P_k(s_0, s) P_0^k(s)$$

is possible only when  $P_0(s) = 1$ .

Kendall obtained, for the branching birth-and-death process (the process with  $f(s, x) = p_0(s) + p_1(s)x + p_2(s)x^2$ ), a necessary and sufficient condition for degeneration <sup>(2)</sup>. In this case equation (1) is integrated in elementary functions. In the present note it is shown that Kendall's condition is sufficient for degeneration in the general case and necessary under certain restrictions; also obtained are some limit theorems for the birth-and-death process.

## § 1. Conditions for degeneration.

Put

$$\lim_{x \uparrow 1} \frac{f(v, x)}{x - 1} = a(v).$$

This limit, finite or infinite, always exists. We shall assume that  $a(v)$  is Riemann integrable on every finite interval  $(s, t) \subset [0, \infty)$ .

In this case  $F(s, t, x)$  satisfies the integral equation:

$$F(s, t, x) = 1 - \frac{1 - x}{e^{-\rho(s,t)} + (1 - x) \int_s^t \frac{\beta}{2}(v, F(v, t, x)) e^{-\rho(s,v)} dv}, \quad (2)$$

$$\rho(s, t) = \int_s^t a(v) dv, \quad \frac{\beta}{2}(v, y) = \frac{f(v, y) - a(v)(y - 1)}{(y - 1)^2}.$$

Equation (1) is obtained from (2) by differentiation. Fulfillment of the initial condition is obvious.

Let us note that the process under consideration will be regular. (A process is called regular if, for any  $s$  and  $t$  ( $0 \leq s \leq t$ ),  $\sum_{n=0}^{\infty} P_n(s, t) = 1$ . Passing to the limit in (2) as  $x \uparrow 1$ , we obtain  $\sum_{n=0}^{\infty} P_n(s, t) = 1$ .)

**Theorem 1.** The divergence of the integral

$$\int_s^{\infty} p_0(v) e^{-\rho(s,v)} dv$$

is a sufficient condition for extinction of the nonhomogeneous branching process for which  $a(v)$  is Riemann integrable on every finite interval  $(s, t) \subset [0, \infty)$ .

**Proof.** From (2), putting  $x = 0$ , we find:

$$P_0(s, t) = 1 - \frac{1}{e^{-\rho(s,t)} + \int_s^t \frac{\beta}{2}(v, P_0(v, t)) e^{-\rho(s,v)} dv}. \quad (3)$$

Replacing  $P_0(v, t)$  by 0 on the right-hand side of (3), we obtain

$$P_0(s, t) \geq 1 - \frac{1}{e^{-\rho(s,t)} + \int_s^t (a(v) + p_0(v)) e^{-\rho(s,v)} dv} = 1 - \frac{1}{1 + \int_s^t p_0(v) e^{-\rho(s,v)} dv}.$$

Consequently, if the integral

$$\int_s^{\infty} p_0(v) e^{-\rho(s,v)} dv$$

diverges, then  $\lim_{t \rightarrow \infty} P_0(s, t) = 1$ .

**Theorem 2.** If  $f(s, x)$  is a polynomial in  $x$ , then the condition

$$\int_s^{\infty} p_0(v) e^{-\rho(s,v)} dv = \infty \quad (4)$$

is necessary and sufficient for extinction.

**Proof.** Sufficiency follows from Theorem 1. We shall prove necessity. Suppose the process becomes extinct. Replacing  $P_0(v, t)$  by one in (3), we obtain the estimate

$$P_0(s, t) \leq 1 - \frac{1}{e^{-\rho(s, t)} + \int_s^t \frac{1}{2} b(v) e^{-\rho(s, v)} dv},$$

where

$$b(v) = \sum_{k=2}^n k(k-1) p_k(v)$$

( $n$  is the degree of the polynomial). It follows that

$$\lim_{t \rightarrow \infty} \left[ e^{-\rho(s, t)} + \int_s^t \frac{1}{2} b(v) e^{-\rho(s, v)} dv \right] = \infty.$$

If

$$\lim_{t \rightarrow \infty} \int_s^t b(v) e^{-\rho(s, v)} dv = \infty,$$

then also

$$\lim_{t \rightarrow \infty} \int_s^t (a(v) + p_0(v)) e^{-\rho(s, v)} dv = \infty,$$

since

$$\int_s^t (a(v) + p_0(v)) e^{-\rho(s, v)} dv = \int_s^t \sum_{k=2}^n (k-1) p_k(v) e^{-\rho(s, v)} dv \geq \frac{1}{n} \int_s^t \sum_{k=2}^n k(k-1) p_k(v) e^{-\rho(s, v)} dv.$$

And then from the identity

$$1 + \int_s^t p_0(v) e^{-\rho(s, v)} dv = e^{-\rho(s, t)} + \int_s^t (a(v) + p_0(v)) e^{-\rho(s, v)} dv \quad (5)$$

it follows that

$$\int_s^\infty p_0(v) e^{-\rho(s, v)} dv = \infty.$$

If, however,

$$\lim_{t \rightarrow \infty} \int_s^t b(v) e^{-\rho(s, v)} dv < \infty,$$

then there exists

$$\lim_{t \rightarrow \infty} e^{-\rho(s, t)} = \infty,$$

and then from (5) it follows that

$$\int_s^\infty p_0(v)e^{-\rho(s,v)} dv = \infty,$$

since

$$a(v) + p_0(v) = \sum_{k=2}^n (k-1)p_k(v) \geq 0.$$

If  $f(s, x)$  is not a polynomial, then, under some restrictions, condition (4) remains necessary. Three cases are possible: 1) the limit  $\mathbf{M}\mu_{st}$  as  $t \rightarrow \infty$  does not exist, 2)  $\lim_{t \rightarrow \infty} \mathbf{M}\mu_{st}$  exists and is finite, 3)  $\lim_{t \rightarrow \infty} \mathbf{M}\mu_{st}$  is infinite.

Let us show that in the first case condition (4) is satisfied and the process becomes extinct. Suppose that

$$\int_s^\infty p_0(v)e^{-\rho(s,v)} dv < \infty.$$

Then from (5) it follows that there exists a finite

$$\lim_{t \rightarrow \infty} e^{-\rho(s,t)};$$

and hence there exists the limit  $\mathbf{M}\mu_{st} = e^{\rho(s,t)}$ . A contradiction has been obtained. Consequently,

$$\int_s^\infty p_0(v)e^{-\rho(s,v)} dv = \infty,$$

and by Theorem 1 the process becomes extinct.

Consider the second case. We first show that the divergence of the integral

$$\int_s^\infty p_0(v) dv$$

is a necessary condition for extinction. Note that in the branching process with  $f(s, x) = p_0(s) - p_0(s)x$  the probability of extinction is equal to

$$1 - \exp \left[ - \int_s^\infty p_0(v) dv \right].$$

In the process under consideration, a particle existing at time  $s$  dies simultaneously with the particle of the auxiliary process (with  $f(s, x) = p_0(s) - p_0(s)x$ ), while preservation of the particle or its transformation into a set of particles corresponds to preservation of the particle of the auxiliary process. Consequently, the divergence of the integral

$$\int_s^\infty p_0(v) dv$$

is a necessary condition for extinction of the process. And since in the case under consideration

$$\lim_{t \rightarrow \infty} e^{-\rho(s,t)} = \delta(s) > 0,$$

there exists such a  $t_0$  that

$$\int_{t_0}^t p_0(v) e^{-\rho(s,v)} dv \geq \frac{\delta(s)}{2} \int_{t_0}^t p_0(v) dv.$$

Consequently,

$$\int_s^\infty p_0(v) e^{-\rho(s,v)} dv$$

diverges.

Thus, with the aid of Theorems 1 and 2 and the remarks just made, it is not possible to resolve the question of extinction of a process for which

$$\int_s^\infty p_0(v) e^{-\rho(s,v)} dv < \infty, \quad \lim_{t \rightarrow \infty} \mathbf{M}\mu_{st} = \infty$$

and  $f(s, x)$  is not a polynomial in  $x$ .

**§ 2. Limit theorems.** For homogeneous branching processes the form of the limiting distribution depends on the asymptotics of  $M^*(t)$ —the conditional mathematical expectation of the number of particles under the condition that the process has not become extinct by time  $t$ . When  $M^*(t)$  tends to a constant as

$t \rightarrow \infty$ , the limiting distribution is discrete; if  $M^*(t) \xrightarrow[t \rightarrow \infty]{} \infty$ , then the limiting distribution is continuous, and for an extinct process an exponential distribution is obtained. For an inhomogeneous process  $M^*(s, t) = M\{\mu_{st} \mid \mu_{st} > 0\}$  as  $t \rightarrow \infty$  may tend to a finite limit, to infinity, or have no limit. If  $M^*(s, t)$  tends to a constant, then in this case a discrete distribution is also obtained.

We note that  $M^*(s, t)$  may tend to a constant in the case of a non-extinct process. For a homogeneous process in the non-extinct case the limiting distribution is always continuous.

If  $M^*(s, t) \xrightarrow{t \rightarrow \infty} \infty$ , then for an inhomogeneous process with  $f(s, x) = a(s)(x - 1) + \frac{b(s)}{2}(x - 1)^2$  it can be shown that the distribution

$$S_{st}(y) = \mathbf{P} \left\{ \frac{\mu_{st}}{M^*(s, t)} < y \mid \mu_{st} > 0 \right\}$$

converges to a limiting one; moreover, if the process becomes extinct, then the limiting distribution is exponential.

It is not difficult to obtain the following theorems.

**Theorem 3.** For a non-extinct process with  $f(s, x) = \sum_{k=0}^n p_k(s)x^k$ , the random variable  $\xi_{st} = \frac{\mu_{st}}{M\mu_{st}}$  converges in probability to  $\xi_s$ , whose characteristic function satisfies the functional equation:

$$\varphi(s, \tau e^{\rho(s, t)}) = F(s, t, \varphi(s, \tau))$$

for all  $s, t$  ( $0 \leq s \leq t < \infty$ ), and the partial differential equation (if  $\varphi(s, \tau)$  is continuous in  $s$ )

$$\tau a(s) \frac{\partial \varphi(s, \tau)}{\partial \tau} = \frac{\partial \varphi(s, \tau)}{\partial s} + f(s, \varphi(s, \tau)).$$

If, in addition,  $\lim_{t \rightarrow \infty} M\mu_{st} < \infty$  (this limit exists in the case of a non-extinct process), then  $\mu_{st} \xrightarrow[t \rightarrow \infty]{p} \mu_s$ , which has a discrete distribution concentrated at integer points.

**Theorem 4.** For an extinct process with  $f(s, x) = a(s)(x - 1) + \frac{b(s)}{2}(x - 1)^2$ , the following assertions hold:

1) If as  $t \rightarrow \infty$  there exists a finite limit  $M^*(s, t)$ , then

$$\mathbf{P}\{\mu_{st} = n \mid \mu_{st} > 0\} \xrightarrow[t \rightarrow \infty]{} P_n, \quad \sum_{n=1}^{\infty} P_n = 1,$$

and the limiting distribution does not depend on  $s$ .

2) If  $\lim_{t \rightarrow \infty} M^*(s, t) = \infty$ , then

$$S_{st}(y) = \mathbf{P} \left\{ \frac{\mu_{st}}{M\mu_{st}} < y \mid \mu_{st} > 0 \right\} \xrightarrow{t \rightarrow \infty} S(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y}, & y \geq 0. \end{cases}$$

Theorem 3 is proved in the same way as the analogous theorem in <sup>(1)</sup>. The proof of Theorem 4 is carried out by means of characteristic functions, which are written out explicitly.

In the case where the limit  $M^*(s, t)$  as  $t \rightarrow \infty$  does not exist, the distribution  $S_{st}(y)$  and the conditional distribution of  $\mu_{st}$  under the condition of non-extinction cannot have a limit, since along different subsequences the characteristic functions tend to different limits.

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*Note: Figure translations are in progress. See original paper for figures.*

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