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## Abstract

## Full Text

## E. R. Tsekanovskii

# On Model Elements of Non-Self-Adjoint Operators

*(Presented by Academician S. L. Sobolev on 16 X 1961)*

**1°.** In the papers ([1],[2]) the question of reducing operators of class $(\Omega)$ to triangular form was studied. It is not difficult to see that an operator $A$ of class $(\Omega)$ can be represented as the coupling of two operators $A_1$ and $A_2$, where $A_1$ is a complete* operator, while $A_2$ is an operator whose spectrum consists only of real points. To reduce the operator $A_1$ to triangular form it is sufficient to construct an orthonormal basis $\psi_1, \psi_2, \dots$ by successive orthogonalization of eigen- and associated elements.

The triangular model of the operator $A_2$ is the continual analogue of a triangular matrix. In this latter case, in a Hilbert space one cannot indicate a basis analogous to the basis $\psi_1, \psi_2, \dots$.

In the present paper generalized elements of a Hilbert space are introduced. In doing so, to each operator of the class under consideration there is put in correspondence a basis consisting of ordinary or generalized elements, with respect to which the operator has the form of a triangular model. From what was said above it is clear that it is sufficient to study operators with purely real spectrum.

**2°.** A bounded linear operator $A$, acting in a Hilbert space $H$, will be assigned to the class $\Omega$ if it has the following properties:

1) the entire spectrum of the operator lies on the real axis;
2) the rank of non-Hermiticity of the operator is equal to $r$ $(r < \infty)$;
3) $A$ is a simple operator, i.e. $H$ coincides with the closure of the linear span of all vectors of the form

$$A^n f \left( n = 0, 1, \dots,\ f \in \frac{A - A^*}{i} H \right).$$

Let us extend $H$ to some Hilbert space $\widetilde{H} = H \oplus H_0$. An operator $\widetilde{A}$, acting in $\widetilde{H}$, is called a simple extension of the operator $A$ ([3]) if the subspaces $H$ and $H_0$ are invariant for it and if $\widetilde{A}$ generates in $H$ the operator $A$, and in $H_0$ some Hermitian operator. Every operator satisfying conditions 1) and 2), but not satisfying condition 3), is a simple extension of an operator of class $\Omega$. Let $A \in \Omega$. Consider the Hilbert space $L_2^{(r)}$, whose elements are matrices

$f(x) = \|f_1(x), f_2(x), \dots, f_r(x)\|$, defined on $[0, l]$, where $l$ is the sum of the moduli of the eigenvalues of the operator $A$. We define the scalar product by the formula

$$(f(x), g(x)) = \int_0^l f(x) g^*(x)\, dx.$$

Then, as is known ($^1$), some simple extension $\widetilde{A}$ of the operator $A$ is unitarily equivalent to the operator acting in $L_2^{(r)}$

$$\widetilde{B} f(x) = \alpha(x) f(x) + i \int_x^l f(t) \pi(t)\, dt\, J \pi^*(x),$$

where $\alpha(x)$ is a bounded nondecreasing function; $\pi(x) = \|\pi_{ij}(x)\|$ is a square matrix of order $r$; $J = \|\delta_{ij}\|$ is a diagonal matrix, each element of whose main diagonal is equal either to $+1$ or to $-1$, i.e.

$$\widetilde{A} = U^{-1} \widetilde{B} U.$$

It is also known $^{(3)}$ that if $A \in \Omega$, then there exists such a simple extension $\widetilde{A}$ of it that

$$\widetilde{A} = \int_0^l \alpha(\lambda)\, dE_\lambda + i \int_0^l E_\lambda \frac{\widetilde{A} - \widetilde{A}^*}{i}\, dE_\lambda, \tag{1}$$

where $E_\lambda$ is a continuous orthogonal resolution of the identity. Let us also note that if $F_\lambda$ is the projection operator in $L_2^{(r)}$ which assigns to the vector $h(t)$ the vector

$$F_\lambda h(t) = \begin{cases} h(t), & 0 \le t \le \lambda, \\ 0, & \lambda < t \le l, \end{cases}$$

then, as shown in $^{(3)}$,

$$E_\lambda = U^{-1} F_\lambda U. \tag{2}$$

3°. Denote by $\{\Psi\}$ the set of vector-functions from $L_2^{(r)}$ all of whose elements are infinitely differentiable functions on the interval $[0, l]$. Introduce in the set $\{\Psi\}$ the system of norms

$$\|f(x)\|_p = \max_{i,\ x \in [0,l]} \{\, |f_i(x)|,\ |f_i'(x)|, \dots,\ |f_i^{(p)}(x)|\, \}$$

$$(i = 1, \dots, r;\ p = 1, 2, \dots).$$

It is easy to see that the norms introduced are comparable and compatible $^{(4)}$.

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* An operator is called complete if it possesses a complete system of eigen- and associated elements.

The set $\{\Psi\}$, with the norms specified in it, is a complete countably normed space $^{(4)}$. Let $U$ be an isometric mapping of the space $\widetilde{H}$ onto the Hilbert space $L_2^{(r)}$, under which the operator $\widetilde{A}$ passes into its model $\widetilde{B}$.

Consider

$$\{\Phi\} = U^{-1}\{\Psi\}.$$

Introduce in the linear space $\Phi$ a system of norms, setting

$$\|f\|_p = \|f(x)\|_p \qquad (f(x) = Uf,\ p = 1, 2, ...).$$

It is not difficult to see that $\Phi$ is a complete countably normed perfect $^{(4)}$ space and that

$$\Phi \subset \widetilde{H} \subset \Phi',$$

where $\Phi'$ is the space of linear continuous functionals defined on $\Phi$. We shall call the space $\Phi$ the space of basic elements, and its elements basic. We shall say that every linear continuous functional $x(\varphi) \in \Phi'$ is generated by a generalized element $x$, and we shall denote the value of the functional on the basic element $\varphi \in \Phi$ by $(\varphi, x)$. Let $x_1$ and $x_2$ be generalized elements. By $a_1 x_1 + a_2 x_2$ we denote the generalized element generating the functional

$$\overline{a_1}(\varphi, x_1) + \overline{a_2}(\varphi, x_2).$$

The linear space of generalized elements obtained in this way will be identified with $\Phi'$.

In what follows, by $x_\lambda$ ($\lambda \in [0, l]$) we shall mean a generalized vector of order $r$, all components of which are generalized elements generated by linear continuous functionals. Using the notion of generalized elements introduced, we shall show that a certain simple extension $\widetilde{A}$ of the operator $A \in \Omega$ has a system of generalized vectors $x_\lambda$ ($\lambda \in [0, l]$) satisfying the conditions:

1) for every $f \in \widetilde{H}$

$$f = \int_0^l f(\lambda)\, x_\lambda\, d\lambda;$$

2) if

$$\widetilde{A}f = \int_0^l g(\lambda)\, x_\lambda\, d\lambda \qquad (g = \widetilde{A}f),$$

then the coefficients $f(\lambda)$ and $g(\lambda)$ are connected with each other by the triangular model, i.e.

$$g(\lambda) = \alpha(\lambda) f(\lambda) + i \int_\lambda^l f(t)\pi(t)\, dt\, J\pi^*(\lambda).$$

The generalized vectors $x_\lambda$ ($\lambda \in [0,l]$) satisfying the conditions listed above will be called **model** vectors.

**Theorem.** *Some simple extension $\widetilde{A}$ of an operator $A \in \Omega$ possesses a system of generalized model vectors $x_\lambda$ ($\lambda \in [0,l]$), orthogonal and complete in the sense that for every element $f \in \widetilde{H}$ the equalities*

$$f = \int_0^l f(\lambda) x_\lambda \, d\lambda, \qquad \|f\|^2 = \int_0^l f(\lambda) f^*(\lambda) \, d\lambda \tag{3}$$

*hold. The integral sums of the integral (3) converge to $f$ strongly.*

**Proof.** Consider the functional

$$(\varphi, f_\lambda^{(i)}) = (\varphi, E_\lambda e_i) \qquad (\varphi \in \Phi;\ i = 1, 2, \dots, r),$$

where $E_\lambda$ is the continuous orthogonal resolution of the identity corresponding to the operator $A$; $e_i(t) = Ue_i$; $e_i(t) = \|0, \dots, 0, 1, 0, \dots, 0\|$, $e_i \in \widetilde{H}$; $i = 1, 2, \dots, r$. Using equality (2), one can show that the functional $(\varphi, f_\lambda^{(i)})$ ($i = 1, \dots, r$; $\lambda \in [0,l]$) has strongly bounded variation $^{(4)}$, and therefore, by a known theorem $^{(4)}$, it is weakly differentiable with respect to the parameter $\lambda$, i.e.

$$(\varphi, x_\lambda^{(i)}) = \frac{d}{d\lambda}(\varphi, E_\lambda e_i) \qquad (\varphi \in \Phi;\ i = 1, \dots, r;\ \lambda \in [0,l]).$$

Let us show that the functional $(\varphi, x_\lambda^{(i)})$ belongs to the space $\Phi'$. Indeed,

$$(\varphi, x_\lambda^{(i)}) = \frac{d}{d\lambda}(\varphi, E_\lambda e_i) = \frac{d}{d\lambda}(U^{-1}\varphi(t),\, U^{-1}F_\lambda U e_i) =$$

$$= \frac{d}{d\lambda}(\varphi(t), F_\lambda e_i(t)) = \frac{d}{d\lambda}\int_0^\lambda \varphi_i(t)\, dt = \varphi_i(\lambda). \tag{4}$$

Equality (4) proves our assertion.

Now let $\varphi \in \Phi$, $f \in \widetilde{H}$. Then

$$(\varphi, f) = \sum_{i=1}^r \int_0^l \varphi_i(\lambda)\, \overline{f_i(\lambda)}\, d\lambda.$$

In view of (4), the last equality may be rewritten in the form

$$(\varphi, f) = \sum_{i=1}^r \int_0^l \overline{f_i(\lambda)}\, (\varphi, x_\lambda^{(i)})\, d\lambda = \sum_{i=0}^r \int_0^l (\varphi, f_i(\lambda) x_\lambda^{(i)})\, d\lambda =$$

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$$= \int_0^l \left( \varphi, \sum_{i=1}^r f_i(\lambda) x_\lambda^{(i)} \right) d\lambda = \int_0^l (\varphi, f(\lambda) x_\lambda) \, d\lambda. \tag{5}$$

By virtue of the perfection of the space $\Phi$ (strong and weak convergence coincide), the integral sums converge strongly. Equalities (3) follow

now from (5). Suppose further that

$$\widetilde{A} f = \int_0^l g(\lambda) x_\lambda \, d\lambda \qquad (g = \widetilde{A} f, \ g(\lambda) = Ug).$$

Then

$$g(\lambda) = \frac{d}{d\lambda} (\widetilde{A} f, E_\lambda e),$$

where

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_r \end{pmatrix}.$$

Since $\widetilde{A} = U^{-1} \widetilde{B} U$, we have

$$\begin{aligned} g(\lambda) &= \frac{d}{d\lambda} (U^{-1} \widetilde{B} U f, \ U^{-1} E_\lambda U e) = \frac{d}{d\lambda} (\widetilde{B} f(t), E_\lambda e(t)) \\ &= \frac{d}{d\lambda} \left\{ \int_0^\lambda a(t) f(t) \, dt + i \int_0^\lambda \left[ \int_t^l f(\xi) \pi(\xi) \, d\xi \ J \pi^*(t) \right] dt \right\} \\ &= a(\lambda) f(\lambda) + i \int_\lambda^l f(t) \pi(t) \, dt \ J \pi^*(\lambda), \end{aligned}$$

where

$$e(t) = Ue = \begin{Vmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \end{Vmatrix}.$$

The theorem is proved.

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*Note: Figure translations are in progress. See original paper for figures.*

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