

# THE STRUCTURE OF AN $\backslash(n\backslash)$ -DIMENSIONAL PARALLELOHEDRON OF THE FIRST TYPE

The present note is devoted mainly to a brief proof of two theorems.

1962

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.13289>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**MATHEMATICS**

**S. S. RYSHKOV**

## **THE STRUCTURE OF AN $n$ -DIMENSIONAL PARALLELOHEDRON OF THE FIRST TYPE**

*(Presented by Academician I. M. Vinogradov on 10 VII 1962)*

The present note is devoted mainly to a brief proof of two theorems.

**Theorem 1.** *Any  $n$ -dimensional primitive parallelohedron of the first type can be transformed into any other  $n$ -dimensional primitive parallelohedron of the first type by means of an affine transformation and stretching of zones of edges<sup>1</sup>.*

**Theorem 2.** *Any  $(n - 1)$ -dimensional face of an  $n$ -dimensional primitive parallelohedron of the first type  $\Pi^n$  is an affine image of the direct product of two primitive parallelehedra of the first type  $\Pi^k$  and  $\Pi^{n-k-1}$ , where the number  $k$  may take any integer values from zero to  $n - 1$ , and a point is taken as the zero-dimensional parallelohedron.*

1°. By a primitive parallelohedron of the first type we shall conventionally mean a primitive parallelohedron  $\Pi$  which can be obtained from the primitive Dirichlet domain of a lattice of the first type (2) by means of an affine transformation and stretching of zones of edges.

Our theorems are also valid for the classical definition of a primitive parallelohedron of the first type as simply an affine image of the primitive Dirichlet domain of a lattice of the first type. The latter assertion is valid only by virtue of Voronoi' s difficult theorem that any primitive parallelohedron is an affine image of the Dirichlet domain of some lattice.

Everywhere in what follows, for brevity of writing, instead of the words “ $n$ -dimensional primitive parallelohedron of the first type” we shall write “parallelohedron  $\Pi$ ” or  $\Pi^n$ , if it is necessary to emphasize the dimension.

2°. **Proof of Theorem 1.** Let  $\Pi_1$  and  $\Pi_2$  be two  $n$ -dimensional parallelehedra situated in Euclidean space  $E^n$ . Without restricting the generality of the argument, one may assume that they are Dirichlet domains of certain lattices. Let also

$$\mathfrak{Z}_1 = (a_1, a_2, \dots, a_{n+1})$$

<sup>1</sup>The proof of this theorem in the three-dimensional case, due to Zhitomirskii, is included in the article (1).

be a reduced Selling frame  $(\mathfrak{Z}_1)$  of the first lattice, and

$$\mathfrak{Z}_2 = (b_1, b_2, \dots, b_{n+1})$$

a reduced Selling frame of the second lattice. Through the midpoints of the vectors  $a_i$ , where  $i = 1, 2, \dots, n+1$ , of the first frame draw the  $(n-1)$ -dimensional planes  $E_i^{n-1}$  perpendicular to them. The resulting set, consisting of  $(n+1)$  planes, bounds an  $n$ -dimensional simplex  $T_1$ , which contains the parallelohedron  $\Pi_1$ . Let us note the fact that to each zone of edges of the parallelohedron  $\Pi_1$  there corresponds an edge of the simplex  $T_1$  parallel to it, and conversely, to each edge of the simplex  $T_1$  there corresponds a zone of parallel and mutually equal edges of the parallelohedron  $\Pi_1$ . Indeed, each edge  $\gamma$  of the simplex  $T_1$  is the intersection of  $(n-1)$  planes  $E_i^{n-1}$ ; since the plane  $E_i^{n-1}$  is perpendicular to the vector  $a_i$ , the edge  $\gamma$  is perpendicular to the  $(n-1)$ -dimensional plane spanned by  $(n-1)$  vectors from the frame  $\mathfrak{Z}_1$ . But, as follows from the study of decompositions of the first type, produced—

Voronoi, such planes and the zones of edges perpendicular to them are in one-to-one correspondence.

Let us affinely map the simplex  $T_2$ , constructed from the parallelohedron  $\Pi_2$  and the frame  $\mathfrak{Z}_2$ , onto the simplex  $T_1$ . Under this mapping, by virtue of the parallelism of the edges of the simplex and of the parallelohedron, each zone of edges of the parallelohedron  $\Pi_2$  will pass into a zone of edges parallel to the edges of some zone of edges of the parallelohedron  $\Pi_1$ . By a certain refinement of the evidently established one-to-one correspondence between the vertices of the image  $\Pi'_2$  of the parallelohedron  $\Pi_2$  and the vertices of the parallelohedron  $\Pi_1$ , one can show that the individual polyhedral angles at the vertices of one parallelohedron can be made to coincide with the individual polyhedral angles at the corresponding vertices of the other parallelohedron by means of a parallel translation.

Now let us stretch the zones of edges of the parallelohedron  $\Pi'_2$  so that each of its edges coincides in length with the edge of the parallelohedron  $\Pi_1$  parallel to it. Perform such a parallel translation of the resulting parallelohedron  $\Pi''_2$  that one of its vertices coincides with the corresponding vertex of the parallelohedron  $\Pi_1$ . It is obvious that, by the preceding remarks, the parallelohedron  $\Pi''_2$  thus translated coincides with the parallelohedron  $\Pi_1$ .

**3°. Lemma 1.** *Let a compact set  $A$  lie in Euclidean space  $E^p$ , and let a compact set  $B$  lie in the space  $E^q$ . Denote by  $E^{p+q} = E^p \times E^q$  the direct product of the spaces  $E^p$  and  $E^q$ . Then the convex hull of the set  $A \times B$ , lying in the space  $E^{p+q}$ , is the direct product of the convex hulls of the sets  $A$  and  $B$ .*

**4°.** A Zelling frame  $\mathfrak{Z} = (a_1, a_2, \dots, a_{n+1})$  will be called a regular Zelling frame if all scalar products of the form  $a_i a_j$  for  $i \neq j$  are equal to one another. It is obvious that in each Euclidean space  $E^n$  there exists, up to similarity and motion, only one regular Zelling frame.

The Dirichlet domain of a lattice whose reduced Zelling frame is a regular Zelling frame will henceforth, for brevity, be called a regular parallelohedron (of the corresponding dimension). In the two-dimensional case this is a regular hexagon; in the three-dimensional case it is the well-known cuboctahedron of Archimedes, bounded by six squares and eight regular hexagons.

The following lemma, which is very important for what follows, is proved by direct calculation (one considers the equations of the  $(n - 1)$ -dimensional faces of the regular parallelohedron and of some of their intersections). It would be desirable to find a simple purely geometric proof of it.

**Lemma 2.** *The  $(n - 1)$ -dimensional face of a regular  $n$ -dimensional parallelohedron perpendicular to an arbitrary vector of the regular Zelling frame associated with it is an  $(n - 1)$ -dimensional regular parallelohedron.*

5°. The information collected in this subsection about the structure of the vertices and faces of a regular parallelohedron can be found in another form in the paper <sup>(2)</sup>.

By a snake  $\langle a_{i_1}, a_{i_2}, \dots, a_{i_{n+1}} \rangle$  of the collection of vectors  $\mathfrak{Z} = (a_1, a_2, \dots, a_{n+1})$ , whose sum is zero, we shall mean a closed polygonal line with links  $a_{i_1}, a_{i_2}, \dots, a_{i_{n+1}}$ . We denote the beginning of the first link by  $O$  and shall assume that all snakes under consideration begin at the point  $O$ . If any  $n$  vectors of the system  $\mathfrak{Z}$  are linearly independent, then the convex hull of the snake  $\langle a_{i_1}, a_{i_2}, \dots, a_{i_{n+1}} \rangle$  is an  $n$ -dimensional simplex  $L^n(a_{i_1}, \dots, a_{i_{n+1}})$ . In what follows, for simplicity, we shall assume that the original system  $\mathfrak{Z}$  is a regular Zelling frame.

It follows from Voronoi's investigation that the simplexes  $L^n(a_{i_1}, \dots, a_{i_{n+1}})$ , constructed on all possible snakes of the Zelling frame  $\mathfrak{Z}$ , are the simplexes  $L$  and together form the star of the point  $O$  in the decomposition  $(L)$ . The centers of the balls,

circumscribed about these simplexes, are the vertices of a regular  $n$ -dimensional parallelohedron  $D$ , having its center at the point  $O$ .

Let us now divide the set  $\mathfrak{Z}$  into  $k + 1$  groups of vectors  $1 : a_{i_1}, a_{i_2}, \dots, a_{i_r}; 2 : a_{i_{r+1}}, a_{i_{r+2}}, \dots, a_{i_{r+p}}; \dots; (k + 1) : a_{i_{r+p+\dots+q+1}}, a_{i_{r+p+\dots+q+2}}, \dots, a_{i_{n+1}}$  and consider the chain

$\langle a_{i_1} + \dots + a_{i_r}, a_{i_{r+1}} + \dots + a_{i_{r+p}}, \dots, a_{i_{r+p+\dots+q+1}} + \dots + a_{i_{n+1}} \rangle$ . Obviously, this chain determines a  $k$ -dimensional face  $L^k$  of each simplex of the set  $(L^n)^*$ —the set of simplexes constructed on chains whose first  $r$  links are the vectors of the first group, the next  $p$  links are the vectors of the second group, etc. It turns out that in the  $(n - k)$ -dimensional plane perpendicular to the simplex  $L^k$ , intersecting it at the center of the circumscribed sphere, there lies an  $(n - k)$ -dimensional face of the parallelohedron  $D$ , and all vertices of this face are the centers of the spheres circumscribed about the simplexes of the set  $(L^n)^*$ .

In the same way one can also describe the faces of faces (naturally, not only of

the highest dimensions) of the parallelohedron  $D$ .

6°. **Lemma 3.** *Every  $(n - 1)$ -dimensional face of an  $n$ -dimensional regular parallelohedron  $D^n$  is the direct product of two regular parallelohedra of smaller dimensions.*

We shall prove this lemma by induction: for  $n = 3$  the lemma is obvious. Suppose it is true for all natural numbers less than  $n$ , and prove it for  $n$ .

It follows easily from the assertion of the lemma that any face of dimension  $k$  of the parallelohedron  $D^{n-1}$  is the direct product of several regular parallelohedra of smaller dimensions. Each such product of faces has a number of vertices equal to  $(n_1 + 1)!(n_2 + 1)! \cdots (n_p + 1)!$ , where

$$\sum_{i=1}^p n_i = k,$$

the  $n_i$  denote the dimensions of the factors, and  $p$  the number of factors. Since the number of vertices of a regular  $k$ -dimensional parallelohedron is equal to  $(k + 1)!$ , and under our conditions

$$(k + 1)! > (n_1 + 1)!(n_2 + 1)! \cdots (n_p + 1)!,$$

we may assert that any  $k$ -dimensional face of an  $(n - 1)$ -dimensional regular parallelohedron having  $(k + 1)!$  vertices is a regular  $k$ -dimensional parallelohedron.

Take an arbitrary  $(n - 1)$ -dimensional face  $\Gamma$  of the polyhedron  $D^n$  and write down the corresponding chain. This chain obviously has the form:

$$\langle a_{i_1} + a_{i_2} + \cdots + a_{i_r}, a_{i_{r+1}} + a_{i_{r+2}} + \cdots + a_{i_{n+1}} \rangle.$$

It is not difficult to see that the faces  $\Gamma^{r-1}$  and  $\Gamma^{n-r}$  of the polyhedron  $D^n$ , determined by the chains

$$\langle a_{i_1}, a_{i_2}, \dots, a_{i_r}, a_{i_{r+1}} + a_{i_{r+2}} + \cdots + a_{i_{n+1}} \rangle$$

and

$$\langle a_{i_1} + a_{i_2} + \cdots + a_{i_r}, a_{i_{r+1}}, a_{i_{r+2}}, \dots, a_{i_{n+1}} \rangle$$

are at the same time faces of the polyhedron  $\Gamma$ . At the same time these faces are faces of those  $(n - 1)$ -dimensional faces of the polyhedron  $D^n$  which, by Lemma 2, are regular parallelohedra. On the basis of what has been said, since the polyhedra  $\Gamma^{r-1}$  and  $\Gamma^{n-r}$  have respectively  $r!$  and  $(n - r + 1)!$  vertices, it may be concluded that these faces are regular parallelohedra of their respective dimensions. Since the regular Zellingner frame is taken as the basis of all constructions, it is easy to see that the planes carrying the faces  $\Gamma^{r-1}$  and  $\Gamma^{n-r}$  are perpendicular, and their product forms the  $(n - 1)$ -dimensional plane  $E^{n-1}$  carrying the face  $\Gamma$ . At the same time, from consideration of the corresponding chains it may be concluded that the set of vertices of the face  $\Gamma$  coincides, up

to parallel translation, with the product of the sets of vertices of the faces  $\Gamma^{r-1}$  and  $\Gamma^{n-r}$ , and the face  $\Gamma$  itself, by Lemma 1, with the product  $\Gamma^{r-1} \times \Gamma^{n-r}$ .

7°. For the proof of Theorem 2 it is now sufficient, on the basis of Theorem 1, to transform the regular parallelohedron into a prescribed and,

since over the faces likewise only an affine transformation and stretching of the edges are performed, they remain parallelohedra of the first type.

8°. Denote by  $\Pi$  an arbitrary  $n$ -dimensional parallelohedron of the first type, and by  $D$  a regular  $n$ -dimensional parallelohedron. Describe around the polytope  $D$  (the Dirichlet domain) a simplex in the same way as was done in 2°, and mark the faces of the parallelohedron  $D$  lying in the faces of the simplex. Then transform, on the basis of Theorem 1, the parallelohedron  $D$  into the parallelohedron  $\Pi$ .

Some faces, which are obviously primitive parallelohedra, of the polytope  $\Pi$  have turned out to be marked. If we now draw the  $(n-1)$ -dimensional planes containing the marked faces of the polytope  $\Pi$ , then we obtain a simplex containing it, which in what follows we shall call the simplex circumscribed about the parallelohedron  $\Pi$ . It is clear that two simplexes can be circumscribed about every parallelohedron, and these simplexes are centrally symmetric to one another.

**Theorem 3.** Denote by  $T$  the simplex circumscribed about the  $n$ -dimensional parallelohedron  $\Pi$ . Let  $T_1, T_2, \dots, T_k$  be such faces of the simplex  $T$  that no two of them have common vertices and every vertex of the simplex  $T$  is a vertex of one of the simplexes  $T_i$ . Then the parallelohedron  $\Pi$  has  $k!$  faces of dimension  $(n-k+1)$ , which are affine images of products of primitive parallelohedra  $\Pi_1, \Pi_2, \dots, \Pi_k$ , in such a way that the simplexes  $T_1, T_2, \dots, T_k$  are homothetic to the simplexes circumscribed about the images of the parallelohedra-factors.

**Example 1.** Let all the simplexes  $T_i$  be points; then the faces asserted by the theorem are the vertices of the parallelohedron  $\Pi$ .

**Example 2.** Let  $n = 2k - 1$  and all the simplexes  $T_i$  be segments. Such a partition can be made in  $(2k)!/k!2^k$  ways; this means that a  $(2k-1)$ -dimensional parallelohedron has  $(2k)!/2^k$  parallelepipedal faces of dimension  $k$  and has no parallelepipedal faces of larger dimension.

**Example 3.** Let  $T_1$  be an arbitrary vertex of the simplex  $T$ , and  $T_2$  the face opposite this vertex; then there exists a pair of  $(n-1)$ -dimensional faces of the parallelohedron  $\Pi$  which are primitive parallelohedra inscribed in the simplex homothetic to the simplex  $T_2$ , and the number of such pairs is  $n+1$ .

**Example 4.** Let  $T_1$  be an arbitrary edge of the simplex  $T$ , and  $T_2$  the face opposite this edge. This means that there exist two faces of the parallelohedron  $\Pi$  which are prisms over a primitive  $(n-2)$ -dimensional parallelohedron whose circumscribed simplex is homothetic to the simplex  $T_2$ , and whose generating line of the prism is parallel to the edge  $T_1$ . The number of such pairs is equal to  $(n+1)n/2$ .

The proof of Theorem 3 is based on the consideration of various zonotopes constructed on the regular Zelling edge, and on the application of Theorem 1.

9°. The following formula is valid

$$\sum_{i=1}^{\frac{n(n+1)}{2}} \frac{\alpha_i}{\gamma_i} = 1,$$

where by  $\alpha_i$  are denoted the lengths of the noncongruent edges of the parallelohedron  $\Pi$ , and by  $\gamma_i$  the lengths of the respectively parallel edges of the simplex circumscribed about the parallelohedron.

V. A. Steklov Mathematical Institute  
Academy of Sciences of the USSR

Received  
5 VII 1962

### CITED LITERATURE

1. B. Delaunay, *Izv. AN SSSR*, No. 1 and 2 (1929).
2. G. F. Voronoi, *Collected Works*, 2, Kiev, 1952, pp. 239–368.
3. B. N. Delone, *UMN*, vol. 3, 16 (1937).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*