MATHEMATICS

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1962

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Abstract

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MATHEMATICS

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TWO THEOREMS ON INFINITE-DIMENSIONAL SPACES

(Presented by Academician P. S. Aleksandrov, 23 XI 1961)

1. A normal space X is called **weakly infinite-dimensional** if, for every countable system of pairs of closed sets (A_i, B_i), $A_i \cap B_i = \emptyset$, one can find a system of closed sets C_i (the set C_i will be called a partition between A_i and B_i) such that each C_i separates the corresponding sets A_i and B_i and $\bigcap_i C_i = \emptyset$; otherwise the space X is called strongly infinite-dimensional.*

Theorem 1. *Under every continuous mapping f of a weakly infinite-dimensional bicompactum X onto a strongly infinite-dimensional bicompactum Y, there is in Y a point y whose complete inverse image $f^{-1}y$ has cardinality not less than the cardinality of the continuum.*

In other words, if, under a continuous mapping f of a weakly infinite-dimensional bicompactum X onto a bicompactum Y, the complete inverse images of all points of Y have cardinality less than the continuum, then the bicompactum Y is also weakly infinite-dimensional.

An analogous theorem for another class of infinite-dimensional compacta was proved in ([6]), namely, instead of weakly (and, respectively, strongly) infinite-dimensional spaces, countable-dimensional (and, respectively, non-countable-dimensional) compacta were considered there.** As B. T. Levshenko showed ([3]), every countable-dimensional space is weakly infinite-dimensional. The question of whether the converse is true, i.e. whether the classes of weakly infinite-dimensional and countable-dimensional compacta coincide or not, was posed by P. S. Aleksandrov (see ([1]), p. 54, or ([2]), p. 14) and remains open to this day. In the case that these classes coincide, Theorem 1 will turn out to be a generalization of the result from ([6]) cited above to bicompact spaces.***

Proof of Theorem 1. It is enough to prove the following lemma.

Lemma 1. *Under the conditions of Theorem 1, there exist in the space X two disjoint closed sets X_0 and X_1 such that $fX_0 = fX_1 = Y_1$, and the set Y_1 is strongly infinite-dimensional.*

Indeed, suppose that Lemma 1 has been proved. Then, by induction, one can construct a decreasing sequence of strongly infinite-dimensional closed subsets Y_n in Y, and in X a system of closed subsets $X_{i_1 \dots i_n}$ $(i_k = 0, 1)$ such that

$X_{i_1 \dots i_n} \cap X_{j_1 \dots j_n} = \emptyset$, if $(i_1 \dots i_n) \neq (j_1 \dots j_n)$, $X_{i_1 \dots i_n} \subset X_{i_1 \dots i_{n-1}}$, and $f X_{i_1 \dots i_n} = Y_n$. If this is done, then, as is easy to verify, any point $y \in \bigcap_n Y_n$ will have a complete

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* This definition was proposed by P. S. Aleksandrov (see ([1]), p. 54, or ([2]), p. 14). Yu. M. Smirnov indicated another variant of the definition of weak (and, respectively, strong) infinite-dimensionality (see ([3,5])), namely: a space is called weakly infinite-dimensional in the sense of Yu. M. Smirnov if, for every countable system of pairs of closed sets (A_i, B_i), $A_i \cap B_i = \emptyset$, one can find such a finite system of partitions C_i, $i \leq k$ (where k depends on the system of pairs under consideration), between A_i and B_i, that $\bigcap_{i \leq k} C_i = \emptyset$. For compact spaces these two definitions are equivalent.

** A space is called countable-dimensional if it is the sum of a countable number of zero-dimensional ($\dim R_i = 0$) sets R_i. W. Hurewicz proved that a separable space possessing a complete metric is countable-dimensional if and only if it has transfinite dimension ([2]).

*** A generalization of another kind (to metric spaces) was obtained by Yu. M. Smirnov in ([7]).

preimage of cardinality not less than the continuum. Thus, suppose that the sets Y_k and $X_{i_1 \dots i_k}$, $k = 1, \dots, n-1$, have already been constructed. Applying Lemma 1 to the mapping $f : X_{0 \cdots 0} \to Y_{n-1}$, we find in $X_{0 \cdots 0}$ two disjoint closed subsets $X'_{0 \cdots 00}$ and $X'_{0 \cdots 01}$ such that $f X'_{0 \cdots 00} = f X'_{0 \cdots 01} = Y'_n \subset Y_{n-1}$, and the set Y'_n is strongly infinite-dimensional. Then we apply Lemma 1 to the sets Y'_n and $X_{0 \cdots 1} \cap f^{-1} Y'_n$; we obtain the sets Y''_n, $X'_{0 \cdots 10}$, and $X'_{0 \cdots 11}$. Repeating this procedure 2^{n-1} times, we finally obtain a strongly infinite-dimensional closed subset $Y_n \subset Y_{n-1}$ and sets $X'_{i_1 \dots i_n}$ such that $X'_{i_1 \dots i_{n-1} i_n} \subset X_{i_1 \dots i_{n-1}}$. Put then

$$X_{i_1 \dots i_n} = f^{-1} Y_n \cap X'_{i_1 \dots i_n},$$

and the systems of subsets in X and Y that we need have been constructed.

Proof of Lemma 1. Since the bicompactum Y is strongly infinite-dimensional, in Y there exists a countable system of pairs of closed sets (A_i, B_i), $A_i \cap B_i = \emptyset$, which cannot be separated by partitions with empty intersection.* Let (A'_i, B'_i) be the system of pairs of closed sets in X which are the complete preimages of the sets A_i, B_i. Since the bicompactum X is weakly infinite-dimensional, for some finite subsystem (A'_i, B'_i), $i = 1, \dots, n$, there will be partitions C'_i in X with empty intersection. Let M, N be closed sets in X such that $A'_1 \subset M$, $B'_1 \subset N$, $M \cap N = C'_1$, $M \cup N = X$. Let, further, $C = f C'_1$ and $D = fM \cap fN$. The set D is a partition in Y between A_1 and B_1. Since in Y the system of pairs (A_i, B_i), $i = 1, 2, \dots$, cannot be separated by partitions with empty intersection, in the bicompactum D the following system is inseparable by partitions with empty intersection:

$$(A_i \cap D,\ B_i \cap D), \quad i = 2, 3, \dots$$

(see the proof of Lemma 5 from ([5])). Therefore the bicompactum D is strongly infinite-dimensional.

The proof of the lemma is by induction on the number n. If $n = 1$, then $C'_1 = M \cap N = \emptyset$, and we may put

$$Y_1 = D, \qquad X_0 = f^{-1}D \cap M, \qquad X_1 = f^{-1}D \cap N.$$

Suppose now that the lemma is true if the number of "separated" pairs in X is less than n. First case: the pairs

$$(A_i \cap C,\ B_i \cap C), \quad i = 2, 3, \dots,$$

are inseparable on the set C by partitions with empty intersection. Hence the bicompactum C is strongly infinite-dimensional. Consider then the mapping f on the set C'_1. The complete preimages of the pairs $(A_i \cap C,\ B_i \cap C)$, $i \geq 2$, in C'_1 are the pairs

$$(A'_i \cap C'_1,\ B'_i \cap C'_1).$$

The sets $C'_i \cap C'_1$, $i = 2, \dots, n$, are partitions for the first $n - 1$ pairs of this system, and moreover

$$\bigcap_{i=2}^{n}(C'_i \cap C'_1) = \bigcap_{i=1}^{n} C'_i = \emptyset.$$

Therefore, by the induction hypothesis, in C and C'_1 there will be found sets satisfying the conditions of the lemma. Consider the remaining case: in C there exist partitions E_i, $i = 2, \dots, m$, between the sets $A_i \cap C$ and $B_i \cap C$ such that

$$\bigcap_{i=2}^{m} E_i = \emptyset.$$

We may suppose that each E_i has type G_δ. Then in D there exist partitions F_i between the sets $A_i \cap D$ and $B_i \cap D$, $i = 2, \dots, m$, such that $F_i \cap C = E_i$. Since the pairs $(A_i \cap D,\ B_i \cap D)$, $i = 2, 3, \dots$, cannot be separated in D by partitions with empty intersection, the set

$$F = \bigcap_{i=2}^{m} F_i$$

is strongly infinite-dimensional (see again the proof of Lemma 5 from ([5])). Moreover, $F \cap C = \emptyset$. Therefore we may put

$$Y_1 = F, \qquad X_0 = f^{-1}F \cap M, \qquad X_1 = f^{-1}F \cap N.$$

This completes the proof of the lemma, and with it also the proof of Theorem 1.**

* We shall constantly use this convenient expression (or its modifications), which in the present case means that there are no partitions C_i between A_i and B_i such that

$$\bigcap_i C_i = \emptyset.$$

** The proof of Lemma 1 given above remains valid if X and Y are normal spaces, respectively weakly and strongly infinite-dimensional in the sense of Yu. M. Smirnov, and the mapping f is closed.

Remark 1. Theorem 1 can be extended to the noncompact case as follows:

Theorem 1′. *For every closed continuous mapping f of a weakly infinite-dimensional (in the sense of Yu. M. Smirnov) complete metric space with a countable base X onto a strongly infinite-dimensional (in the sense of P. S. Aleksandrov)* normal countably paracompact space Y, there is a point y in Y whose full preimage has cardinality not less than the cardinality of the continuum.*

For the proof of the theorem the following modification of Lemma 1 is used:

Lemma 1′. *Let f be a closed mapping of a weakly infinite-dimensional (in the sense of Yu. M. Smirnov) normal space X onto a strongly infinite-dimensional (in the sense of P. S. Aleksandrov) countably paracompact space Y, and let ω be a countable closed covering of the space X. Then in X there are two disjoint closed sets X_0 and X_1, contained in some elements of the covering ω, such that $fX_0 = fX_1 = Y_1$, and the closed set Y_1 in Y is strongly infinite-dimensional.*

First, repeating with minor changes the arguments from the proof of Lemma 1, we construct in X disjoint closed sets X_0' and X_1' such that the set $Y_0' = fX_0' = fX_1'$ is strongly infinite-dimensional (in the sense of P. S. Aleksandrov). Let

$$X_0' = \bigcup_{i=1}^{\infty} X_0^i,$$

where each X_0^i is contained in some element of the covering ω. Then

$$Y_1' = \bigcup_i fX_0^i,$$

and since Y_1' is strongly infinite-dimensional, for some i the set $Y_1'' = fX_0^i$ is strongly infinite-dimensional. Repeating the analogous arguments for Y_1'' and $X_1' \cap f^{-1}Y_1''$, we construct the required sets Y_1, X_0, X_1.

The proof of Theorem 1′ proceeds analogously to the proof of Theorem 1, with the sole change that the sets $X_{i_1 \cdots i_n}$ must have diameter less than $1/n$.

Remark 2. With the help of arguments analogous to those used above, one can give a simple proof of the following theorem of K. Morita (see ([4])):

Let f be a closed $(m+1)$-fold mapping of a normal space X onto a normal space Y. Then

$$\dim Y \leq \operatorname{Ind} X + m.$$

Let $\operatorname{Ind} X \leq n$ and $\dim Y \geq n+m$; we shall show that then the multiplicity of the mapping f is not less than $m+1$. We prove this by induction on n and m. If $\operatorname{Ind} X = -1$ or if $m = 0$, then the assertion is true. Suppose that the assertion is true if $\operatorname{Ind} X \leq n-1$, or if $\operatorname{Ind} X \leq n$ and $\dim Y \geq n+m-1$. Let $\operatorname{Ind} X \leq n$ and $\dim Y \geq n+m$. In the space Y there are $n+m$ pairs of closed sets (A_i, B_i), $A_i \cap B_i = \emptyset$, which cannot be separated by partitions with empty intersection. Further we use the notation from the proof of Lemma 1; moreover, since $\operatorname{Ind} X \leq n$, one may assume that $\operatorname{Ind} C_1' \leq n-1$. If the pairs $(A_i \cap C, B_i \cap C)$, $i = 2, \dots, n+m$, cannot be separated in C by partitions with empty intersection, then, according to the induction hypothesis (using the fact that $\operatorname{Ind} C_1' \leq n-1$), the mapping f already on the set C_1' has multiplicity $\geq m+1$. If, however, these pairs can be separated in C by partitions with empty intersection, then the pairs $(A_i \cap D, B_i \cap D)$ can be separated in D by partitions F_i, whose intersection F does not meet C. The set F is nonempty, since the pairs $(A_i \cap D, B_i \cap D)$ cannot be separated in D by partitions with empty intersection (cf. the corresponding place in the proof–

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This is important: there exists a strongly infinite-dimensional space Y in the sense of Yu. M. Smirnov (the discrete sum of a countable number of cubes of increasing dimension), onto which the zero-dimensional space X (the sum of a countable number of Cantor sets) can be mapped so that the mapping is closed and the full preimage of each point is finite.

of Lemma 1). Let U be a closed neighborhood of the set F in D, not intersecting C. Then in the set D there exist pairs of disjoint closed sets (K_i, L_i), $i = 2, \dots, n+m$, such that $A_i \cap D \subset K_i$, $B_i \cap D \subset L_i$, and $\bigcap_{i=2}^{n+m} \{D \setminus (K_i \cap L_i)\} \subset U$. Since the intersection of any partitions for the pairs (K_i, L_i) lies in U, and since these partitions are at the same time also partitions for the pairs $(A_i \cap D, B_i \cap D)$, the $n+m-1$ pairs of closed sets $(K_i \cap U, L_i \cap U)$ cannot be separated in U by partitions with empty intersection; consequently, $\dim U \geq n+m-1$. On the other hand, since $U \cap C = \emptyset$, in X there are two disjoint closed sets $f^{-1}U \cap M$ and $f^{-1}U \cap N$, each of which is mapped onto U. Since $\dim U \geq n+m-1$, and $\operatorname{Ind}(f^{-1}U \cap M) \leq n$, by the induction hypothesis there is in U a point having in $f^{-1}U \cap M$ at least m preimages; moreover, this point has at least one preimage in $f^{-1}U \cap N$. Consequently it has at least $m+1$ preimages in X, as was required to prove.

2. B. T. Levshenko proved [3] the following proposition:

A normal space X is strongly infinite-dimensional in the sense of Yu. M. Smirnov if and only if there exists a mapping f of the space X into the Hilbert parallelepiped Q such that the following condition is satisfied:

B. *For every finite-dimensional face F in Q and projection $\pi : Q \to F$, the mapping πf of the space X into the cube F is essential.*

Theorem 2. *A bicompactum X is strongly infinite-dimensional if and only if there exists a mapping f of the bicompactum X into the Hilbert parallelepiped Q such that the following condition is satisfied:*

G. *For every finite-dimensional face F in Q, the mapping f of the set $f^{-1}F$ into the cube F is essential.*

Theorem 2 follows, obviously, from the following proposition:

Theorem $2'$**.** *For every mapping f of a bicompactum X into Q, conditions B and G are equivalent.*

Proof. Obviously, condition B follows from condition G. We shall show that the converse is true. Let the mapping $f = \{f_1, \dots, f_n, \dots\}$ satisfy condition B. As B. T. Levshenko showed [3], this is equivalent to saying that the system of pairs (A_i, B_i), $A_i = \{x : f_{ix} = 0\}$, $B_i = \{x : f_{ix} = 1\}$, cannot be separated in X by partitions with empty intersection. Suppose that, onto some finite-dimensional face $F = \{q \in Q : q_i = 0,\ i > n\}$, its full preimage $\Phi = f^{-1}F$ is mapped inessentially. Then on the set Φ there are functions g_i', $i = 1, \dots, n$, such that they map Φ into the boundary S of the cube F and coincide with the functions f_i on the set $f^{-1}S$. Construct on the bicompactum X functions g_i, $i = 1, \dots, n$, whose values lie between zero and one, such that $g_i = g_i'$ if $x \in \Phi$; $g_i = 0$ if $x \in A_i$, and $g_i = 1$ if $x \in B_i$. Put $g_i = f_i$ if $i \geq n + 1$. Consider the mapping $g = \{g_1, \dots, g_n, \dots\}$ of the bicompactum X into Q, and the pairs (A_i', B_i'), $A_i' = \{x : g_{ix} = 0\}$, $B_i' = \{x : g_{ix} = 1\}$, of closed sets in X. For every i we have $A_i \subset A_i'$, $B_i \subset B_i'$, and therefore the system of pairs (A_i', B_i') cannot be separated in X by partitions with empty intersection. Since X is a bicompactum, it follows from this that g must map X onto the whole Hilbert parallelepiped Q. On the other hand, by the construction of g, it does not cover the point $(1/2, 1/2, \dots, 1/2, 0, 0, \dots)$. The theorem is proved.

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Received
21 XI 1961

REFERENCES

1. P. S. Aleksandrov, *UMN*, **6**, No. 5 (45), 43 (1951).
2. V. Gurevich, G. Wallman, *Dimension Theory*, Moscow, 1948.
3. B. T. Levshenko, *Vestn. Mosk. Univ.*, ser. matem., No. 5, 219 (1959).
4. K. Morita, *Proc. Japan. Acad.*, **32**, No. 3, 161 (1956).
5. E. G. Sklyarenko, *Izv. AN SSSR, ser. matem.*, **23**, No. 2, 197 (1959).

6. E. G. Sklyarenko, *DAN*, **126**, No. 6, 1203 (1959).

7. Yu. M. Smirnov, *DAN*, **141**, No. 4 (1961).

Note: Figure translations are in progress. See original paper for figures.