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MATHEMATICS

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SOLUTION OF SOME ALGORITHMIC PROBLEMS FOR COMMUTATIVE SEMIGROUPS

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Many algorithmic problems are undecidable for groups and for semigroups. In considering the simplest case—commutative semigroups—it is natural to expect algorithmic decidability of most mass problems.

The algorithmic decidability of the word problem is known ([1,2]) for finitely defined commutative semigroups, i.e., commutative semigroups specified by a finite number of generators and defining relations. In this note an algorithm is constructed that makes it possible to determine whether a finitely defined commutative semigroup is a group; and certain other problems are also listed for which solution algorithms have been obtained.

Let the system of defining relations of a finitely defined commutative semigroup $\mathfrak{S}$ with generators

$$a_1, a_2, \dots, a_n \tag{1}$$

have the form

$$A_i = B_i \quad (i = 1, 2, \dots, s), \tag{2}$$

where A_i, B_i are nonempty words in $\mathfrak{S}$.

If in the system (2) there is a defining relation of the form

$$a_i = N \quad \text{or} \quad N = a_i, \tag{3}$$

where the word N does not contain the generator a_i, then this defining relation may be deleted from (2), replacing a_i everywhere by the word N, and thereby obtaining an isomorphic semigroup with a smaller number of generators.

The elements of our semigroup are words of the form $a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}$, where $p_1, p_2, \dots, p_n$ are nonnegative integers. In writing an arbitrary word, the zero power of a generator will be omitted. We shall assume that not all the exponents $p_1, p_2, \dots, p_n$ can be equal to zero simultaneously, i.e., by a word of the semigroup we shall always mean a nonempty word.

Transformations of any word of the form A_iX into the word B_iX, and conversely, where X is an arbitrary word of the semigroup $\mathfrak{S}$, will be called elementary and will be represented in the form of schemes $A_iX \to B_iX$, $B_iX \to A_iX$. We shall also say that the elementary transformation $A_i \to B_i$ ($B_i \to A_i$) is applicable to the word A_iX (B_iX). If the word B_i (A_i) contains generators that are not in A_i (B_i), then we shall say that the transformation $A_i \to B_i$ ($B_i \to A_i$) generates new generators in A_iX (B_iX). We shall agree to adjoin to the transformation schemes also schemes of the form $X \to X$, where X is an arbitrary word of the semigroup. These transformations will be called tautological.

By an elementary transformation we shall also mean the replacement of A_i by B_i (or conversely) after a preliminary permutation of the generators in order to obtain A_i (or B_i) at the beginning of the word being transformed.

The following lemmas are almost obvious.

Lemma 1. *A finitely defined commutative semigroup $\mathfrak{S}$ with generators (1) is a group if and only if in $\mathfrak{S}$ there exists*

word E, which is the identity of the semigroup and contains all generators, i.e.

$$E = a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}, \quad \text{where } p_i > 0 \ (i = 1, 2, \dots, n).$$

Lemma 2. Suppose that in the system of defining relations (2) of the finitely defined commutative semigroup $\mathfrak{G}$ with generators (1) there are no defining relations of the form (3). Then, if $\mathfrak{G}$ is a group, the system (2) must contain defining relations of the form $a_i = a_{iA}$ or $a_{iA} = a_i$, at least one for each a_i $(i = 1, 2, \dots, n)$.

Lemma 3. Suppose that the word E, which is the identity of the finitely defined commutative semigroup $\mathfrak{G}$, does not contain some of the generators. Moreover, suppose that E has the property that every elementary transformation $A_i \to B_i$, $B_i \to A_i$ $(i = 1, 2, \dots, s)$ is either applicable to some power of the word E, but does not generate new generators, or is not applicable to any power of the word E. Then the semigroup $\mathfrak{G}$ is not a group.

The lemmas listed above make it possible to formulate an algorithm Φ for recognizing whether the semigroup $\mathfrak{G}$, given by the system of generators (1) and defining relations (2), is a group.

1. The k-th step $(1 \le k \le s)$ consists in taking the defining relation $A_k = B_k$ and seeing whether it has the form (3). If not, one proceeds to the next step of Φ. If it does, then, replacing a_i by the word N in the remaining

defining relations, one removes the relation $A_k = B_k$ from the system (2) and proceeds to the next step of Φ. After carrying out s such steps, one obtains a semigroup $\mathfrak{G}'(\cong \mathfrak{G})$, given by the generators

$$a_{i_1}, a_{i_2}, \dots, a_{i_p} \tag{4}$$

and the defining relations

$$A_i' = B_i' \quad (i = 1, 2, \dots, m). \tag{5}$$

2. The $(s+j)$-th step $(1 \le j \le p-1)$ consists in checking whether the system (5) contains at least one defining relation of the form

$$a_{i_j} = a_{i_j} A \quad \text{or} \quad a_{i_j} A = a_{i_j}. \tag{6}$$

If not, then the process stops, and (on the basis of Lemma 2) $\mathfrak{G}$ is not a group. If it does, one proceeds to the next step of Φ.

3. The $(s+p)$-th step consists in checking whether (5) contains at least one defining relation of the form $a_{i_p} = a_{i_p} A$ or $a_{i_p} A = a_{i_p}$. If not, then the process breaks off, and $\mathfrak{G}$ is not a group. If it does, then all defining relations of the form (6) are written down for each j $(j = 1, 2, \dots, p)$:

$$\begin{aligned} a_{i_1} &= a_{i_1} A_{1r} \quad (r = 1, 2, \dots, n_1), \\ a_{i_2} &= a_{i_2} A_{2r} \quad (r = 1, 2, \dots, n_2), \\ &\dots\dots\dots\dots\dots\dots \\ a_{i_p} &= a_{i_p} A_{pr} \quad (r = 1, 2, \dots, n_p), \end{aligned} \tag{7}$$

the product E of all A_{ir} occurring in (7) is formed, and one proceeds to the next step of Φ.

4. The $(s+p+t)$-th step $(1 \le t \le p)$ consists in deciding whether the equality $a_{i_t} = a_{i_t} E$ holds in $\mathfrak{G}'$. If $a_{i_t} \ne a_{i_t} E$, then the process ends, and $\mathfrak{G}$ is not a group. If $a_{i_t} = a_{i_t} E$, one proceeds to the next step of Φ.
5. The $(s+2p+1)$-th step consists in checking whether the identity E contains all generators (4). If it does, then the process breaks off, and (on the basis of Lemma 1) $\mathfrak{G}$ is a group. If not, one proceeds to the concluding procedure described in the next item.
6. If one of the elementary transformations $A_1' \to B_1'$ or $B_1' \to A_1'$ is applicable to some natural power q of the word E and generates in E^q new generators, then one acts on E^q by this transformation $(E^q \to E_1)$ and removes the relation $A_1' = B_1'$ from (5). Here the number q is chosen so large that under this transformation $E^q \to E_1$ not a single one disappears.

generator occurring in E. If none of the indicated transformations is applicable to any power of the word E, then E is left unchanged. Finally, if the indicated transformations are applicable to E^q, but do not produce new generators, then, leaving E unchanged, one excludes $A'_1 = B'_1$ from (5). The transformation $E \to E_1$ in the last two cases is tautological.

In an analogous way, the resulting word E is tested by the elementary transformations $A'_2 \to B'_2$ and $B'_2 \to A'_2$, and so on up to the last pair of transformations $A'_m \to B'_m$ and $B'_m \to A'_m$. As a result one obtains the word E_m. Using the schemes of the remaining defining relations, one again repeats, in the manner described, the test of the word E_m, and so on. This process stops when either a word E_r is obtained to no positive power of which any elementary transformations of the remaining defining relations are applicable, or when not a single defining relation remains in the system (5).

7. If the word E_r obtained after completion of the procedure of item 6 contains all generators (4), then, on the basis of Lemma 1, $\mathfrak{G}$ is a group. If the word E_r does not contain some of the generators (4), then by Lemma 3 the semigroup $\mathfrak{G}$ is not a group.

Using the constructed algorithm Φ, we easily find an algorithm for recognizing whether the semigroup $\mathfrak{G}$ has an identity, and also an algorithm that makes it possible to establish the presence or absence of an inverse for any element in an arbitrary semigroup $\mathfrak{G}$ with identity.

A semigroup H with zero is called nilpotent if some power k of it is equal to zero:

$$H^k = \underbrace{H \cdot H \cdot \ldots \cdot H}_{k \text{ times}} = 0.$$

The question of the existence of such a number k for a finitely defined commutative semigroup is algorithmically solved by the following

Theorem 1. *In order that a commutative semigroup with zero, given by a system of generators (1) and defining relations (5), be nilpotent, it is necessary and sufficient that the word M_i, obtained as the result of applying to a_i the procedure of item 6 of the algorithm Φ, contain all generators (1) for all $i = 1, 2, \ldots, n$.*

The question of the existence of a zero in a finitely defined commutative semigroup, given by the system of generators (1) and defining relations (2), is solved as follows.

To the system of generators (1) we add new generators

$$a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1} \tag{8}$$

in the same number, and to the defining relations (2) the new relations

$$a_i a_i^{-1} = 1, \quad a_i^{-1} a_i = 1 \quad (i = 1, 2, \dots, n), \tag{9}$$

where 1 denotes the empty word. As is known, the associative system (with commutative multiplication) given by the generators (1) and (8) and defining relations (2) and (9) will be an Abelian group. We shall call this group the group given by the generators (1) and defining relations (2).

Theorem 2. *A commutative semigroup given by generators (1) and defining relations (2) has a zero if and only if the group given by generators (1) and defining relations (2) is the trivial group.*

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Note: Figure translations are in progress. See original paper for figures.

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