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1962

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Abstract

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THE PLANCHEREL MEASURE FOR RIEMANNIAN SYMMETRIC SPACES OF NONPOSITIVE CURVATURE

(Presented by Academician P. S. Aleksandrov, February 21, 1962)

Let G be a connected semisimple Lie group with finite center, and let U be its maximal compact subgroup. Denote by $\mathcal{E}$ the homogeneous space G/U. As is known, $\mathcal{E}$ is a Riemannian symmetric space of nonpositive curvature. In the present paper an explicit form of the Plancherel measure for such spaces is found.

Harish-Chandra showed (^{1}) that the Plancherel measure is closely related to the asymptotics of zonal spherical functions on $\mathcal{E}$, and found an integral representation for the density of this measure. T. S. Bhanu-Murti (^{2}) computed the explicit form of the density of the Plancherel measure for all (except one) classical symmetric spaces of maximal rank. The formulas obtained in (^{2}) made it possible, to a certain extent, to predict the structure of the density of the Plancherel measure for arbitrary symmetric spaces of the type under consideration.

Let $\mathfrak{h}$ be a Cartan subalgebra* of the space $\mathcal{E}$; let Σ be the system of its roots relative to $\mathfrak{h}$; let Σ^+ (Σ^-) be the set of positive (negative) roots for some ordering. Denote by $\mathfrak{z}^+$ ($\mathfrak{z}^-$) the subalgebras spanned by the root vectors corresponding to the positive (negative) roots. By the Iwasawa theorem, every element g of G can be represented in the form $g = z^- h u$ ($z^- \in Z^-$, $h \in H$, $u \in U$). The element h occurring in this decomposition will be denoted by $h(g)$. For every complex-valued linear form χ on $\mathfrak{h}$ set $h^\chi = \exp(\chi(\xi))$, where $\xi = \ln h$ is the preimage of the element h under the canonical mapping of the algebra $\mathfrak{h}$ onto the group H. Using the Cartan scalar product (ξ_1, ξ_2), which is positive definite on $\mathfrak{h}$, we shall identify the form χ with the vector $\chi = \xi + i\eta$ ($\xi, \eta \in \mathfrak{h}$).

Let R be an arbitrary open half-space in $\mathfrak{h}$, bounded by a hyperplane passing through the origin of the coordinates. Put

$$\Sigma_R = \Sigma \cap R, \qquad \Sigma_R^+ = \Sigma^+ \cap R, \qquad \Sigma_R^- = \Sigma^- \cap R.$$

Denote by $\mathfrak{z}_R, \mathfrak{z}_R^+, \mathfrak{z}_R^-$ the subalgebras spanned by the root vectors corresponding to the roots from $\Sigma_R, \Sigma_R^+, \Sigma_R^-$, respectively. Consider the integral**

$$I(\mathcal{E}, R, \nu) = \int h^\chi(z_R^+)\, dz_R^+,$$

where $\chi = -(\rho + \nu)$,

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$$

(in this sum each root occurs as many times as its multiplicity). If Σ_R^+ is empty, then we put $I(\mathcal{E}, R, \nu) = 1$. Denote by Δ_R the set of vectors $\xi \in \mathfrak{h}$ such that $(\xi, \alpha) > 0$ for all $\alpha \in \Sigma_R^+$. It will be shown below that the integral $I(\mathcal{E}, R, \nu)$ converges absolutely if $\operatorname{Re} \nu \in \Delta_R$. In the case when $\Sigma_R^+ = \Sigma^+$, and hence $\mathfrak{z}_R^+ = \mathfrak{z}^+$, we shall denote the integral $I(\mathcal{E}, R, \nu)$ simply by $I(\mathcal{E}, \nu)$, and the set Δ_R by Δ. The convergence of the integral $I(\mathcal{E}, \nu)$ for $\operatorname{Re} \nu \in \Delta$ is proved in (1^a). We note that if the rank of the space $\mathcal{E}$ is equal to 1 and Σ_R^+ is nonempty, then $\Sigma_R^+ = \Sigma^+$, and hence $I(\mathcal{E}, R, \nu) = I(\mathcal{E}, \nu)$.

To each vector $\lambda \in \Delta$ there corresponds a certain unitary representation of the group G, realized in spherical functions on the space $\mathcal{E}$. The regular representation of the group G in L_2 on $\mathcal{E}$ decomposes into a continuous direct sum of the indicated representations. In this case the Plancherel measure has the form $\mu(d\lambda) = |c(\mathcal{E}, \lambda)|^{-2} d\lambda$, where $d\lambda$ is Euclidean measure in Δ and, as follows from (1), $c(\mathcal{E}, \lambda) = I(\mathcal{E}, i\lambda)(I(\mathcal{E}, \rho))^{-1}$. The integral $I(\mathcal{E}, i\lambda)$ diverges. It must be understood as the analytic continuation of $I(\mathcal{E}, \nu)$ in ν. Thus the problem of computing the Plancherel measure reduces to computing the integral $I(\mathcal{E}, \nu)$. We compute an even more general integral $I(\mathcal{E}, R, \nu)$. It turns out that this integral is equal to the product of analogous integrals for certain symmetric spaces of rank 1 associated with the space $\mathcal{E}$. Namely, denote by Σ_0^+ the subset of positive roots which are not integral multiples of other positive roots. With each root $\alpha \in \Sigma_0^+$ we associate a symmetric space $\mathcal{E}_\alpha$ in the following way. Denote by $\mathfrak{G}_\alpha$ the subalgebra generated by all root vectors corresponding to the roots α and $-\alpha$. The subalgebra $\mathfrak{G}_\alpha$ is semisimple. It is invariant under the involutive automorphism which singles out the subalgebra $\mathfrak{U}$ in $\mathfrak{G}$. The subalgebra $\mathfrak{U}_\alpha = \mathfrak{U} \cap \mathfrak{G}_\alpha$ is a maximal compact subalgebra in $\mathfrak{G}_\alpha$. The symmetric space $\mathcal{E}_\alpha = G_\alpha / U_\alpha$ has rank 1. As a Cartan subalgebra of the space $\mathcal{E}_\alpha$ one may take the line $\mathfrak{H}_\alpha$ in $\mathfrak{H}$ on which the root α is situated.*

The principal result of the present paper is the following theorem.

Theorem 1. *The integral* $I(\mathcal{E}, R, \nu)$ *converges absolutely if and only if* $\operatorname{Re} \nu \in \Delta_R$, *and, with a suitable normalization of measures,*

$$I(\mathcal{E}, R, \nu) = \prod_{\alpha \in \Sigma_0^+ \cap R} I(\mathcal{E}_\alpha, \nu_\alpha), \tag{1}$$

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* We agree to denote Lie groups by capital Latin letters, and their Lie algebras by the corresponding lowercase Gothic letters.

** Everywhere in this article, $\int f(p)\, dp$ denotes the integral over the whole group P, where dp is the invariant measure on P.

where ν_α *is the restriction of the form* $(\nu, \mathfrak{H})$ *to the line* $\mathfrak{H}_\alpha$*.*

From Theorem 1 it follows immediately that

Corollary. *The formula*

$$c(\mathcal{E}, \lambda) = \prod_{\alpha \in \Sigma_0^+} c(\mathcal{E}_\alpha, \lambda_\alpha)$$

is valid.

For all spaces $\mathcal{E}$ of rank 1 the integral $I(\mathcal{E}, \nu)$ is easily computed directly.[**] After substituting the resulting expressions for $I(\mathcal{E}_\alpha, \nu_\alpha)$ in (1), we obtain Theorem 2.

Theorem 2. *With a suitable normalization of the measure* dz_R^+ *on the group* Z_R^+*, we have:*

$$I(\mathcal{E}, R, \nu) = \prod_{\alpha \in \Sigma_R^+} \mathrm{B}\left(\frac{p_\alpha}{2}, \frac{p_{\alpha/2}}{4} + \frac{(\nu, \alpha)}{(\alpha, \alpha)}\right),$$

where p_α *is the multiplicity of the root* α*, and* $\mathrm{B}(x, y)$ *is the beta function.*

We shall precede the proof of Theorem 1 by two lemmas.

Lemma 1. *Let a finite-dimensional algebra* $\mathfrak{N}$ *be decomposed into the direct sum of two subspaces* $\mathfrak{A}$ *and* $\mathfrak{B}$*. Suppose that in* $\mathfrak{N}$ *there exists a system of subspaces* $I_0, \dots, I_{l+1}$ *such that* $\mathfrak{N} = I_0 \supset I_1 \supset \cdots \supset I_{l+1} = \{0\}$, $[\mathfrak{N}, I_k] \subset I_{k+1}$*, and* $I_k = \mathfrak{A}_k + \mathfrak{B}_k$*, where* $\mathfrak{A}_k = \mathfrak{A} \cap I_k$, $\mathfrak{B}_k = \mathfrak{B} \cap I_k$*. Choose a basis in the space* $\mathfrak{A}_l$ $(\mathfrak{B}_l)$*, extend it to a basis in* $\mathfrak{A}_{l-1}$ $(\mathfrak{B}_{l-1})$*, and so on. The aggregate of the coordinates of a vector* $\mathfrak{n} \in \mathfrak{N}$ *with respect to the basis vectors,*

lying in $\mathfrak{A}_k(\mathfrak{B}_k)$, but not lying in $\mathfrak{A}_{k+1}(\mathfrak{B}_{k+1})$, by $a_k(n)$, $(b_k(n))$. Then

$$\exp(n) = \exp(a_0) \dots \exp(a_l) \exp(b_l) \dots \exp(b_0)^*,$$

where $a_k \in \mathfrak{A}_k$, $b_k \in \mathfrak{B}_k$ and the coordinates $a_i(a_k)$ $(b_i(b_k))$ for $i \neq k$ are equal to zero. Moreover, if

$$a_k(n) = a_k, \quad b_k(n) = b_k, \quad a_k(\hat{a}_k) = a_k', \quad b_k(\hat{b}_k) = b_k',$$

then

$$a_k' = a_k + f_k(a_0, \dots, a_{k-1}; b_0, \dots, b_{k-1}), \quad b_k' = b_k + \varphi_k(a_0, \dots, a_{k-1}, b_0, \dots, b_{k-1}). \tag{2}$$

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[*] We note that if $\mathcal{E}$ is a space of maximal rank, then the subalgebra $\mathfrak{G}_\alpha$ is a three-dimensional subalgebra associated with the root α.

[**] See also (1^a). We note that the final formula for spaces of rank 1, given in (1^a) (no. 13, p. 303), contains an inaccuracy.

The proof of Lemma 1 is easily obtained, for example, by means of the Campbell-Hausdorff formula.

Remark. Let us note that the conditions of Lemma 1 are satisfied if $\mathfrak{A}$ is an ideal in $\mathfrak{N}$, and $\mathfrak{B}$ is an arbitrary complementary subspace. In this case, as the system of subspaces I_k one may take the subspaces

$$\mathfrak{N} = I_0 \supset \mathfrak{A} \supset [\mathfrak{N}, \mathfrak{A}] \supset [\mathfrak{N}, [\mathfrak{N}, \mathfrak{A}]] \supset \cdots.$$

Moreover, $\mathfrak{B}_k = \{0\}$ for $k \geq 1$.

Lemma 2. Let some set of roots $T \subset \Sigma$ of the space $\mathcal{E}$ be invariant with respect to some root α (i.e. if $\beta \in T$ and $\beta + k\alpha \in \Sigma$, then $\beta + k\alpha \in T$), and let $\delta = \sum_{\beta \in T} \beta$. Then $(\alpha, \delta) = 0$.

Proof. Let S_α be the reflection in the plane perpendicular to the root α. As is known, S_α is an element of the Weyl group of the space $\mathcal{E}$. By virtue of the invariance of T with respect to α, we have $S_\alpha T = T$, since

$$S_\alpha \beta = \beta - 2(\alpha, \beta)(\alpha, \alpha)^{-1} \cdot \alpha.$$

Consequently, $S_\alpha \delta = \delta$. On the other hand,

$$S_\alpha \delta = \delta - 2(\alpha, \delta)(\alpha, \alpha)^{-1} \cdot \alpha,$$

whence $(\alpha, \delta) = 0$.

We now pass to the proof of Theorem 1. First note that formula (1) reduces the assertion on the convergence of $I(\mathcal{E}, R, \nu)$ to the corresponding assertion for a space of rank 1, whose validity is easily established directly. Thus it remains only to prove formula (1). We shall carry out the proof by induction on the number $k(R)$ of vectors contained in $\Sigma_0^+ \cap R$. For $k(R) = 0$ Theorem 1 is obvious. Suppose now that Theorem 1 is valid for all half-spaces Q for which $k(Q) < k(R)$. For the half-space R one can construct such an open half-space Q, bounded by a hyperplane L passing through the origin, that

$$\Sigma_Q^+ \subset \Sigma_R^+, \quad \Sigma_R^+ \setminus \Sigma_Q^+ \subset L$$

and $\Sigma_0^+ \cap L$ consists of one root α. With this root α there is associated, in the manner indicated above, the symmetric space $\mathcal{E}_\alpha = G_\alpha / U_\alpha$ of rank 1 with Cartan subalgebra $\mathfrak{h}_\alpha$. In $\mathfrak{G}_\alpha$ choose subalgebras $\mathfrak{Z}_\alpha^+$ and $\mathfrak{Z}_\alpha^-$, analogous to the subalgebras $\mathfrak{Z}^+$ and $\mathfrak{Z}^-$ of the algebra $\mathfrak{G}$. It is easy to see that

$$\mathfrak{Z}_R^+ = \mathfrak{Z}_Q^+ + \mathfrak{Z}_\alpha^+$$

and $\mathfrak{Z}_Q^-$ is an ideal in $\mathfrak{Z}_R^-$. On the basis of the remark to Lemma 1, every element $z_R^+ \in Z_R^+$ can be represented in the form $z_R^+ = z_Q^+ z_\alpha^+$. In canonical coordinates the invariant measure on the nilpotent Lie group is Euclidean measure. From (2) it follows that

$$dz_R^+ = dz_Q^+ dz_\alpha^+.$$

Therefore

$$I(\mathcal{E}, R, \nu) = \int h^{\chi}(z_Q^+ z_\alpha^+)\, dz_Q^+ dz_\alpha^+.$$

The element z_α^+ belongs to the half-simple group G_α and in it decomposes as:

$$z_\alpha^+ = z_\alpha h_\alpha(z_\alpha^+) u_\alpha.$$

Let us now observe that $[\mathfrak{G}_\alpha, \mathfrak{Z}_Q] \subset \mathfrak{Z}_Q$. Therefore

$$(z_\alpha^-)^{-1} z_Q^+ z_\alpha^- = z_Q(z_Q^+, z_\alpha^+) = z_Q \in Z_Q.$$

Obviously,

$$h^{\chi}(z^- g u) = h^{\chi}(g),$$

where $z^- \in Z^-$ and $u \in U$. Hence

$$I(\mathcal{E}, R, \nu) = \int h^{\chi}(z_{Qh_\alpha}(z_\alpha^+))\, dz_Q^+ dz_\alpha^+.$$

For every closed half-space $P \subset Q$ put $\Xi_P = P \cap \Sigma$. There is only a finite number of different nonempty systems of roots Ξ_P. Number them in decreasing order so that

$$\Sigma_Q = \Xi_0 \supset \Xi_1 \supset \dots \supset \Xi_l.$$

Let I_k be the subalgebra in $\mathfrak{Z}_Q$ spanned by the root vectors corresponding to roots from Ξ_k. We have

$$\mathfrak{Z}_Q = I_0 \supset I_1 \supset \dots \supset I_{l+1} = \{0\}.$$

It is obvious that if I_i corresponds to $\Xi_i = \Xi_{P_i}$, and I_j corresponds to $\Xi_j = \Xi_{P_j}$, and if the half-space P_k is the arithmetic sum of the half-spaces P_i and P_l, then $[I_i, I_j] \subset I_k$, where I_k corresponds to $\Xi_k = \Xi_{P_k}$. Hence it follows that $[\mathfrak{Z}_Q, I_k] \subset I_{k+1}$. Put $\mathfrak{A} = \mathfrak{Z}_Q^-$ and $\mathfrak{B} = \mathfrak{Z}_Q^+$. The subalgebra $\mathfrak{Z}_Q$ is nilpotent. The subspaces $\mathfrak{A}, \mathfrak{B}$, and I_k satisfy the conditions of Lemma 1; moreover, as the basis appearing in its formulation we may choose root vectors. Consequently,

$$z_Q = z_Q\left(z_Q^-, z_\alpha^+\right) = z_Q^- \tilde{z}_Q^+,$$

where $z_Q^- \in Z_Q^-$, $\tilde{z}_Q^+ \in Z_Q^+$. Let us now note that $[\mathfrak{G}_\alpha, I_k] \subset I_k$. Hence it follows that the canonical coordinates $a_k(z_Q) = a_k(\ln z_Q)$ and $b_k(z_Q) = b_k(\ln z_Q)$ are expressed linearly only in terms of the coordinates $b_i(\tilde{z}_Q^+) = b_i(\ln z_Q^+)$, where $i \leq k$, and, by virtue of the nilpotency of the group Z_α^-, the transition matrix from $b_i(z_Q^+)$ to $b_k(z_Q)$ is nilpotent. Therefore (see (2))

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* By $\exp(n)$ we denote the canonical mapping of the algebra $\mathfrak{N}$ onto the corresponding simply connected group N.

$$b_i(\tilde{z}_Q^+) = b_k + \psi_k(b_0, \dots, b_{k-1}),$$

where $b_i = b_i(z_Q^+)$. Thus, $dz_Q^+ = d\tilde{z}_Q^+$. As a result we obtain

$$I(\mathcal{E}, R, \nu) = \int h^\chi\big(z_Q^+ h_\alpha(z_\alpha^+)\big)\, dz_Q^+\, dz_\alpha^+.$$

Next,

$$z_Q^+ h_\alpha(z_\alpha^+) = h_\alpha(z_\alpha^+)\, \hat{z}_Q^+,$$

where

$$\hat{z}_Q^+ = h_\alpha^{-1}(z_\alpha^+)\, z_Q^+ h_\alpha(z_\alpha^+) \in Z_Q^+.$$

The measures dz_Q^+ and $d\hat{z}_Q^+$ are connected by the relation

$$dz_Q^+ = h_\alpha^\vartheta(z_\alpha^+)\, d\hat{z}_Q^+,$$

where $\vartheta = \sum_{\beta \in \Sigma_Q^+} \beta$. Let us now note that, if $h_0 \in H$, then

$$h^\chi(h_0 g) = h_0^\chi \cdot h^\chi(g).$$

Consequently,

$$I(\mathcal{E}, R, \nu) = \int h_\alpha^{\chi+\vartheta}(z_\alpha^+)\, dz_\alpha^+ \cdot \int h^\chi(\hat{z}_Q^+)\, d\hat{z}_Q^+.$$

It is obvious that

$$h_\alpha^{\chi+\vartheta}(z_\alpha^+) = h_\alpha^{\chi'+\vartheta'}(z_\alpha^+),$$

where χ', ϑ' are the projections of the vectors χ and ϑ onto the line $\mathfrak{H}_\alpha$. We have:

$$-\rho + \vartheta = -\rho_\alpha + \frac{1}{2} \sum_{\beta \in \Sigma_Q} \beta,$$

where

$$\rho_\alpha = \frac{1}{2} \sum_{\beta \in \Sigma^+ \cap L} \beta,$$

i.e. ρ_α is the vector appearing in the definition of the integral $I(\mathcal{E}_\alpha, \nu_\alpha)$. The set of roots Σ_Q is invariant with respect to the root α. Therefore, by Lemma 2, the projection of the vector $-\rho + \vartheta$ onto the line $\mathfrak{H}_\alpha$ coincides with the vector $-\rho_\alpha$. The projections of the vectors onto the line $\mathfrak{H}_\alpha$ correspond to the restrictions of the corresponding forms to this line. Thus,

$$I(\mathcal{E}, R, \nu) = I(\mathcal{E}_\alpha, \nu_\alpha)\, I(\mathcal{E}, Q, \nu),$$

and the validity of Theorem 1 for the integral $I(\mathcal{E}, R, \nu)$ follows directly from the induction hypothesis.

Received
14 II 1962

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Note: Figure translations are in progress. See original paper for figures.

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