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Abstract

Full Text

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ON PERIODIC SUBGROUPS OF SOLVABLE MATRIX GROUPS

(Presented by Academician A. I. Mal'cev on 28 V 1962)

Let Γ be a maximal solvable subgroup of the full linear group over an algebraically closed field of arbitrary characteristic. In the present article the conjugacy of maximal periodic subgroups of the group Γ in Γ is proved.

By the letter Δ we shall denote an algebraically closed field of arbitrary characteristic, and by E_n the identity matrix of order n .

§ 1. Lemma 1. *Let Γ be a maximal irreducible primitive solvable subgroup of the full linear group $GL(n, \Delta)$. Then the set Φ of all elements of finite order from Γ forms a normal divisor of the group Γ .*

Proof. Let Δ^* be the multiplicative group of the field Δ , and let

$$\Gamma_0 = \Gamma \cap SL(n, \Delta),$$

where $SL(n, \Delta)$ is the special linear group. Then Γ_0 is a finite group, and moreover

$$\Gamma = \Gamma_0 \Delta^* E_n \quad (1)$$

An element $g = \lambda g_0$, $\lambda \in \Delta^*$, $g_0 \in \Gamma_0$, has finite order if and only if λ belongs to the periodic part P of the group Δ^* . Consequently,

$$\Phi = \Gamma_0 P E_n.$$

The lemma follows from this.

Lemma 2. *Let Γ be a maximal irreducible solvable subgroup of $GL(n, \Delta)$. Then all maximal periodic subgroups Γ are conjugate in Γ .*

Proof. Let Δ^n be the linear space in which Γ acts. Write the complete decomposition of Δ^n into the systems of imprimitivity of the group Γ :

$$\Delta^n = Q_1 \dot{+} Q_2 \dot{+} \dots \dot{+} Q_k.$$

Then, as shown in (2), Δ^n has a basis

$$u_1^1, \dots, u_m^1, \dots, u_1^k, \dots, u_m^k, \quad (1)$$

such that $\Gamma = GT$, the group G consists of all matrices of the form

$$g = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_k \end{bmatrix},$$

where the A_j , independently of one another, run through a maximal primitive solvable subgroup \mathfrak{G} of the group $GL(m, \Delta)$, and the group T is isomorphic to a certain maximal solvable transitive subgroup of the symmetric group S_k ; for $t \in T$,

$$t(u_j^\nu) = u_j^\mu, \quad \nu = 1, \dots, k; \quad j = 1, \dots, m.$$

We add also that u_1^ν, \dots, u_m^ν is a basis of Q_ν .

Let Φ be a maximal periodic subgroup of the group \mathfrak{G} . Then

$$\Psi = \Phi^k T,$$

where Φ^k consists of all matrices of the form

$$\begin{bmatrix} \varphi_1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \varphi_k \end{bmatrix}, \quad \varphi_j \in \Phi,$$

is obviously a periodic subgroup of the group Γ .

We shall show that any periodic subgroup of the group Γ is conjugate in Γ to some subgroup of the group Ψ . Let H be an arbitrary periodic subgroup of the group Γ ; T_1 the group of substitutions effected by H on Q_1, \dots, Q_k , and let Q_1, \dots, Q_l be one of the systems of imprimitivity of the group T_1 . Then in H there are operators h_1, h_2, \dots, h_l such that

$$h_1 = E_n, \quad h_2 Q_1 = Q_2, \dots, \quad h_l Q^1 = Q_l.$$

For h_ν ($\nu = 1, \dots, l$) one may write

$$h_\nu = g_\nu t_\nu, \quad t_\nu \in T, \quad g_\nu \in G,$$

$$t_\nu(u_j^1) = u_j^\nu, \quad j = 1, \dots, m,$$

$$g_\nu = \begin{bmatrix} A_{1\nu} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & A_{k\nu} \end{bmatrix}, \quad A_{i\nu} \in \mathfrak{G}.$$

We now choose a new basis of the space Δ^n :

$$v_1^1, \dots, v_m^1, v_1^2, \dots, v_m^2, \dots, v_1^k, \dots, v_m^k, \quad (2)$$

where $v_j^\nu = u_j^\nu$, if $\nu = 1$ or $\nu > l$,

$$v_j^\nu = h_\nu(u_j^1) = g_\nu(u_j^\nu), \quad \text{if } 1 < \nu \leq l.$$

The matrix expressing the new basis (2) in terms of the old basis (1) has the form

$$c = \begin{bmatrix} E_m & 0 & \cdot & \cdot & 1 \\ 0 & A_{22} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & A_{ll}0 \\ 0 & \cdot & \cdot & \cdot & E_m \end{bmatrix}, \quad (3)$$

where $A_{ij} \in \mathfrak{G}$. Consequently, $c \in G$.

Let h be any operator from H . Then $hQ_\mu = Q_\nu$, $1 \leq \mu, \nu \leq l$. Obviously,

$$\begin{bmatrix} h(v_1^\mu) \\ \cdot \\ \cdot \\ h(v_m^\mu) \end{bmatrix} = B_\mu \begin{bmatrix} v_1^\nu \\ \cdot \\ \cdot \\ v_m^\nu \end{bmatrix}, \quad (4)$$

where B_μ is an $m \times m$ -matrix over Δ .

We shall prove that $B_\mu \in \Phi$. For this, consider $a = h_\mu h_\nu^{-1} h$. Obviously, $a \in H$, $aQ_\mu = Q_\mu$. Hence, from Lemma 1 it follows that the restriction of a to Q_μ belongs to Φ . On the other hand, for a , using (4), one can write

$$\begin{bmatrix} a(v_1^\mu) \\ \cdot \\ \cdot \\ a(v_m^\mu) \end{bmatrix} = B_\mu \begin{bmatrix} h_\mu h_\nu^{-1}(v_1^\nu) \\ \cdot \\ \cdot \\ h_\mu h_\nu^{-1}(v_m^\nu) \end{bmatrix} = B_\mu \begin{bmatrix} v_1^\mu \\ \cdot \\ \cdot \\ v_m^\mu \end{bmatrix}.$$

Consequently, in the basis v_1^μ, \dots, v_m^μ the matrix of the restriction of a to Q_μ coincides with B_μ . In view of (3), in the basis u_1^μ, \dots, u_m^μ the matrix of the restriction of a to Q_μ is equal to $AB_\mu A^{-1}$, where $A \in \mathfrak{G}$. Thus $AB_\mu A^{-1} \in \Phi$. Since Φ is a normal divisor of \mathfrak{G} , it follows that $B_\mu \in \Phi$. We shall now regard the elements of H as matrices. Then, from the fact that $B_\mu \in \Phi$, it follows that the restriction of $c^{-1}Hc$ to

$Q_1 \dot{+} \dots \dot{+} Q_l = L$ is contained in the restriction of Ψ to L (see (3)). Repeating these arguments for each system of imprimitivity of the group T_1 , we obtain

$$d^{-1}Hd \subseteq \Psi, \quad \text{where } d \in G.$$

The lemma follows from this.

§ 2. Theorem. *Let Γ be a maximal solvable subgroup of $GL(n, \Delta)$, where Δ is an algebraically closed field. Then the maximal periodic subgroups of the group Γ are conjugate in Γ .*

Proof. First consider the case where the characteristic of the field Δ is zero. In this case a maximal periodic subgroup of the group Γ is contained in some maximal completely reducible subgroup of the group Γ . As was proved in (3), the maximal completely reducible subgroups of Γ are conjugate in Γ . On the other hand, a maximal completely reducible subgroup D of the group Γ is isomorphic to the direct product of its irreducible components, each of which is a maximal solvable irreducible group (3). By virtue of Lemma 2, the maximal periodic subgroups of the group D are conjugate in D . Consequently, for fields of characteristic zero the theorem is proved.

Let now the characteristic of the field Δ be $p > 0$. Write the matrices of the group Γ in the form

$$g = \begin{bmatrix} a_1(g) & * & \dots & * \\ 0 & a_2(g) & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_k(g) \end{bmatrix}, \quad (5)$$

where $a_j(g)$ runs through a maximal irreducible solvable subgroup Γ_j of the group $GL(n_j, \Delta)$, $n_1 + n_2 + \dots + n_k = n$. Obviously,

$$g = du, \quad (6)$$

where

$$d = \begin{bmatrix} a_1(g) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & a_k(g) \end{bmatrix}, \quad u = \begin{bmatrix} E_{n_1} & * & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & \dots & E_{n_k} & \dots \end{bmatrix}.$$

All matrices u from (6) form an invariant p -subgroup U of the group Γ , while the matrices d form a completely reducible subgroup D . Obviously, the mapping $g \mapsto d$ is a homomorphism φ of the group Γ onto D . If H is a maximal periodic subgroup of the group Γ , then $\varphi(H)$ is a maximal periodic subgroup of D . Consequently, any maximal periodic subgroup H of the group Γ is representable in the form

$$H = VU,$$

where V is a maximal periodic subgroup of the group D . Since the maximal periodic subgroups of the group D are conjugate in D , it follows from (7) and from the invariance of u in Γ that the maximal periodic subgroups of the group Γ are conjugate in Γ . The theorem is proved.

As is known, in $GL(a, \Delta)$ the set of all maximal solvable subgroups splits into only a finite number of classes of conjugate subgroups. Hence, from the theorem just proved there follows

Corollary. The set of all maximal solvable periodic subgroups of $GL(n, \Delta)$ splits into only a finite number of classes of subgroups conjugate in $GL(n, \Delta)$.

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