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**Abstract**

**Full Text**

**MATHEMATICS**

**M. B. KUDAEV**

# INVESTIGATION OF THE BEHAVIOR OF TRAJECTORIES OF SYSTEMS OF DIFFERENTIAL EQUATIONS BY MEANS OF LYAPUNOV FUNCTIONS

*(Presented by Academician I. G. Petrovskii, 11 VII 1962)*

1. We consider a system of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, \dots, x_n), \quad X_i(0, \dots, 0) = 0, \quad i = 1, \dots, n. \quad (1)$$

Here  $X_i(x) \in C^2(U(O))$ ;  $U(O)$  is a neighborhood of the point  $O$ , lying in Euclidean space  $E^n$ ;  $x = (x_1, \dots, x_n)$ ;  $O = (0, \dots, 0)$  is the only singular point in  $U(O)$ .

Refining certain definitions of P. N. Papusha <sup>(1)</sup>, we shall assume that the whole neighborhood  $U(O)$  can be partitioned by topological cones  $H_l$ ,  $l = 1, \dots, k$ , with vertex at the point  $O$ , into  $m$  domains  $\omega_s$ ,  $s = 1, \dots, m$ . In addition, it will sometimes be necessary to use topological cones  $h_r$ , also with vertex  $O$ , which are situated inside the domain  $\omega_s$  (apart from the point  $O$ ). We shall call the latter cones additional cones.

Two formulations of the problem are possible <sup>(2)</sup>: either, as is done in <sup>(1)</sup>, one prescribes such analytic properties of the right-hand sides of equations (1) from which one or another arrangement of the trajectories of system (1) will follow; or, conversely, one prescribes in the domains  $\omega_s$  and on the surfaces  $H_l$  and  $h_r$  the topological arrangement of the trajectories and seeks those analytic conditions which follow from this arrangement.

**Definition 1.** We shall call a sign-constant function  $V(x)$  a **generalized Lyapunov function** if, being nonzero in the domains  $\omega_s$ , it retains one and the same sign in each of them, and if its derivative  $dV/dt$  by virtue of system (1) is a sign-variable function, also retaining its sign in the domains  $\omega_s$ , changing it each time upon passing through a surface  $H_l$ ,  $l = 1, \dots, k$ , while on the surface

$$\bigcup_{l=1}^k H_l$$

$V(x) \geq 0$ ,  $dV/dt = 0$ . Those cones  $H_l$  that enter into the boundary of the domain  $\omega_s$  will be called **adjacent**.

In addition to the first derivative  $F_1 \equiv dV/dt$ , we shall also consider the second derivative  $F_2 \equiv d^2V/dt^2$  of the function  $V(x)$ .

**Definition 2.** Following V. V. Nemytskii, we shall call a system of integral curves **hyperbolic** if all curves of this system are either  $O^+$ - or  $O^-$ -curves having no  $\alpha$ - and  $\omega$ -limit points except the origin (parabolic curves), or curves going off in both directions (hyperbolic curves). The system is called **elliptic** if all integral curves are either parabolic or elliptic, i.e. have as their  $\alpha$ - and  $\omega$ -limit set the singular point  $O$ . Finally, the system is called **elliptic-hyperbolic** if, in it, apart from  $O$ , there are integral curves of all the three classes mentioned (3,4).

II. Let us formulate the following theorems.

**Theorem 1.** *If  $V(x)$  is a generalized Lyapunov function, and the function  $F_2 > 0$  ( $F_2 < 0$ ) on the cones  $H_l$ ,  $l = 1, \dots, k$ , then system (1) will be hyperbolic (elliptic).*

This theorem is a generalization of Theorem 7 from [1], since in the latter constancy of sign of the function  $V(x)$  was not allowed.

Let us note that, for a generalized Lyapunov function  $V(x)$ , simultaneous fulfillment of the relations

$$V(p) = 0, \quad F_2(p) < 0, \quad \text{where } p \in \bigcup_{l=1}^k H_l, \quad p \neq 0,$$

is impossible.

**Theorem 2.** *Suppose:*

- a) *the function  $V(x)$  is positive definite throughout the neighborhood under consideration;*
- b) *the function  $F_1$  is equal to zero on the surface  $\bigcup_{l=1}^k H_l$ , preserves its sign in each domain  $\omega_s$ , and changes it when passing through the surface  $H_l$ ,  $l = 1, \dots, k$ ;*
- c) *the function  $F_2$  preserves its sign on each cone  $H_l$ , but there is at least one pair of adjacent cones forming part of the boundary surfaces of some domain  $\omega_r$ , on which  $F_2$  has opposite signs;*
- d) *there is an additional cone  $h_r$ ,  $h_r \setminus O \subset \omega_s$ , separating this pair of cones and on which the conditions  $F_2 = 0$ ,  $F_1 F_3 > 0$  are satisfied, or there exists a pair of additional cones  $h_{r_i}$ ,  $(h_{r_i} \setminus O) \subset \omega_r$ ,  $i = 1, 2$ , on which the conditions  $F_2 = 0$ ,  $F_1 F_3 < 0$  are satisfied, and the mentioned cones  $h_{r_i}$  are not separated by other additional cones.*

Then system (1) is elliptic-hyperbolic. In this case the domain  $\omega_r$  has an open subset in  $E^n$ , including respectively the surface  $h_r \setminus O$  or the surfaces  $h_{r_i} \setminus O$ , consisting entirely of points of parabolic trajectories.

**Remark.** When speaking of the sign of a function on any of the cones, we never include the singular point in this cone. All the functions considered by us vanish at  $x = 0$ .

III. In this section we shall show that the conditions of Theorems 1 and 2 are, in a certain sense, also necessary.

**Definition 3.** We shall call some domain  $\omega_s$  an **attracting (repelling) hyperbolic domain** if it contains  $O^+-(O^-)$ -curves of system (1), as well as arcs  $f(p, t < 0)$  ( $f(p, t > 0)$ ) of negative (positive) hyperbolic semitrajectories, where  $p$  is an arbitrary (nonzero) point from the union of adjacent cones  $H_l$  corresponding to the domain  $\omega_s$ . The domain  $\omega_s$  is called an **attracting (repelling) elliptic domain** if every trajectory that enters it approaches the singular point  $O$  as  $t \rightarrow +\infty$  ( $t \rightarrow -\infty$ ) and leaves  $\omega_s$  for some finite or infinite  $t < 0$  ( $t > 0$ ), respectively.

**Theorem 3.** Let system (1) be hyperbolic (elliptic), and let each surface  $H_l \setminus O \in C^2(U(O))$ ,  $l = 1, \dots, k$ . Suppose that each hyperbolic (elliptic) curve of system (1) has with the surface

$$\bigcup_{l=1}^k H_l$$

one and only one common point, and that, of any two adjacent domains  $\omega'_s$  and  $\omega''_s$  (whose boundaries have a nonempty intersection distinct from the point  $O$ ), one is attracting if the other is repelling, and conversely.

Then there exists such a sign-definite (positive definite) function  $V(x) \in C^2(U_1(O))$ , where  $U_1(O) \subset U(O)$  is some neighborhood of the point  $O$ , that on

the surface  $\bigcup_{l=1}^k H_l$

$$V(x) = F_1(x) = 0 \quad (V(x) > 0, F_1(x) = 0 \text{ respectively});$$

the product of the functions  $V F_1 < 0$  in each attracting

repelling domain and  $V F_1 > 0$  in each repelling domain  $\omega_s$ ; on the surface  $\bigcup_{l=1}^k H_l$  the function  $F_2 > 0$  ( $F_2 < 0$ , respectively).

We note that for a hyperbolic system (1) satisfying the conditions of Theorem 3, there also exists a positive definite function having all the properties of the function  $V$  of Theorem 3, except, of course, for vanishing at nonsingular points of the surface  $\bigcup_{l=1}^k H_l$ .

In what follows we shall assume that the right-hand sides of equations (1) are functions of class  $C^3(U(O))$ , and also that all  $(n - 1)$ -dimensional cones  $h_r \setminus O$  and  $H_l \setminus O$  are surfaces of class  $C^3(U(O))$ .

**Definition 4.** A domain  $\omega_s$  is called **attracting elliptic-hyperbolic** if every point  $f(p, t)$ , where  $p$  is an arbitrary (different from  $O$ ) point of the union of the adjacent cones  $H_l$  corresponding to the domain  $\omega_s$ , belongs to the domain  $\omega_s$  either only for positive values of  $t$ , or only for negative values of  $t$ ; in the first case, as  $t \rightarrow +\infty$ , it tends to the singular point  $O$  (points of arcs of elliptic semitrajectories), while in the second it leaves  $U(O)$ , no longer intersecting the surface  $\bigcup_{l=1}^k H_l$ , for some finite or infinite  $t < 0$  (points of arcs of hyperbolic semitrajectories); the remaining points of the domain  $\omega_s$  lie on  $O^+$ -trajectories. A **repelling** elliptic-hyperbolic domain is defined analogously.

**Theorem 4.** Suppose that system (1) is elliptic-hyperbolic and that, in the neighborhood under consideration, there is an attracting (repelling) elliptic-hyperbolic domain  $\omega_s$ , which is divided by an additional cone  $h_r$  into two subdomains, one of which contains arcs of all elliptic, and the other arcs of all hyperbolic, semitrajectories; moreover the surface  $h_r \setminus O$  consists of points only of  $O^+$ - ( $O^-$ ) trajectories.

Then in some neighborhood  $U_1(O)$  of the point  $O$ ,  $U_1(O) \subset U(O)$ , there exists a positive definite function  $V(x) \in C^3(U_1(O))$  having the following properties:

- 1)  $F_1 < 0$  ( $F_1 > 0$ ) at points of the domain  $\omega'_s = \omega_s \cap U_1(O)$ ;
- 2)  $F_2 > 0$  at those points of the adjacent cones  $H_l \cap U_1(O)$  corresponding to the domain  $\omega'_s$  through which the hyperbolic semitrajectories pass;  $F_2 < 0$  at those points of these cones through which the elliptic semitrajectories of the domain  $\omega_s$  pass; and at points of the surface  $(h_r \setminus O) \cap \omega'_s$  the conditions

$$F_2 = 0, \quad F_1 F_3 > 0,$$

hold, where  $F_3$  is the third derivative of the function  $V(x)$  along system (1).

With the aid of Theorem 4 it is easy to obtain the converse of Theorem 2 (the case in which, on  $h_r \setminus O$ ,  $F_1 F_3 > 0$ ).

As for the converse of Theorem 2 in the case in which it is assumed there that on two additional cones the conditions  $F_2 = 0$  and  $F_1 F_3 < 0$  hold, this is done with the aid of a theorem similar to Theorem 4. In proving these converse theorems, some ideas and results from the works (5–8) are used.

In conclusion, I take this opportunity to express my deep gratitude to V. V. Nemytskii for posing the problems and for his guidance.

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## CITED LITERATURE

1. P. N. Papush, *Matem. sborn.*, **38**, issue 3 (1956).
2. V. V. Nemytskii, *UMN*, **9**, issue 3 (1954).
3. V. V. Nemytskii, *Vestn. Mosk. univ.*, No. 6 (1960).
4. V. Niemiȳtzki, *Ann. di Mat. pura ed appl.*, **49**, 11 (1960).
5. A. M. Lyapunov, *The General Problem of the Stability of Motion*, 1950.
6. N. N. Krasovskii, *Prikl. matem. i mekh.*, **18**, issue 5 (1954).
7. E. A. Barbashin, *Matem. sborn.*, **29**, issue 2 (1951).
8. I. M. Gelfand, G. E. Shilov, *Generalized Functions and Operations on Them*, Moscow, 1959, p. 182.

*Note: Figure translations are in progress. See original paper for figures.*

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