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Abstract

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THE GROUP OF FOUR-DIMENSIONAL ISOSPIN AND HYPERCHARGE IN AXIOMATIC QUANTUM FIELD THEORY

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Let us suppose that spatial parity is conserved. Let us consider what symmetry group then exists (besides the group of real Lorentz transformations) in axiomatic quantum field theory, based on the quantum-mechanical axioms (I), the axiom of relativistic invariance (II), the axioms of positivity of energy (III), and uniqueness of the vacuum (IV).

1. Since the Wightman functions $W_\varphi\{x\}$, containing physical fields $\varphi_1, \dots, \varphi_n$, must transform under space-time rotations according to M^+-representations of the complex proper Lorentz group $L_+(C)$ ([1]), the operator of reflection of spatial coordinates P_L has the properties $(j, k = 1, 2, 3)$

$$[P_L, M^+_{jk}] = 0, \qquad [P_L, M^-_{\mu\nu}] = 0, \qquad P^+_{LM_{0j}} = -M^+_{0j} P_L \tag{1}$$

(the notation is the same as in ([1])). In consequence of (1), under the reflection P_L the generator $M^+_{1j} + M^-_{0j}$, which produces purely real transformations, goes over into the generator $M^-_{0j} - M^+_{0j}$ for imaginary transformations of the group $L_+(C)$. In view of the restrictions imposed by gradient invariance ([1]), the reflection P_L is impossible in a gradient-invariant theory.

Let us introduce the reflection operator P_T in isospin space, with the properties

$$[P_T, M^-_{jk}] = 0, \qquad [P_T, M^+_{\mu\nu}] = 0, \qquad P^-_{TM_{0j}} = -M^-_{0j} P_T. \tag{2}$$

The operation $P = P_{LP}T$ (“generalized reflection”) preserves the reality of the coordinates and is compatible with gradient invariance.

Thus, in gradient-invariant theories only the generalized reflection has meaning; conservation of spatial parity P_L also implies conservation of isospin parity P_T.

In the absence of unphysical fields A^0_μ and B^0_μ, one may consider P_T and P_L separately (although conservation of P_T is still a consequence of conservation of P_L). Since $P^1_{LA_\mu} P_L^{-1} = -A^1_\mu$ and $PA^0_\mu P^{-1} = -A^0_\mu$, it follows that $P^1_{TA_\mu} P^{-1} = A^1_\mu$, i.e. A^1_μ is an isopseudovector.

2. The group of internal symmetry must possess a Euclidean metric, since otherwise either probabilities will not be invariant with respect to transformations of the group, or, with invariant transition amplitudes, positivity of transition probabilities is lost.

Transformations of the complex group $L_+(C)$ generated by $M^-_{\mu\nu}$ are not connected with the space-time properties of physical fields: in application to $W_\varphi\{z\}$, $[M^-_{\mu\nu}, P_\lambda] = 0$. But the invariants of the $M^-_{\mu\nu}$-transformations have a pseudo-Euclidean metric. Only some of the transformations, namely the subgroup of transformations with the operators M^-_{jk}, $j, k = 1, 2, 3$, are

unitary. This subgroup is the group of rotations in the three-dimensional isospin space $T_j = e_{jik} M^-_{ik}$.

If spatial parity is conserved, an isospin reflection P_T is added to the isospin rotations. The resulting symmetry group with conserved T^2, T_3, P_T is equivalent to the phenomenological symmetry group proposed by d' Espagnat and Prentki ([2]). Since, however, the functions $W_\varphi\{x\}$ are, in addition, invariant with respect to Lorentz-like rotations with M^-_{0k}, the isospinor field u_α must be described not by a two-component isospinor ([2]), but by a bispinor. In the case of a complex (iso-)bispinor, as is known, we automatically obtain conservation of the number of isofermions.

If Γ_μ is an analogue of the Dirac matrices in isospin space and $\Gamma_\mu\Gamma_\nu = i\Sigma_{\mu\nu}$, then an M^-_{jk}-transformation for the bispinor u_α means the substitution $u \to \left(1 + \frac{1}{2}\Sigma_{jk}\omega_{jk}\right) u$, while the reflection P_T means the substitution $u \to i\Gamma_4 u$. Invariance with respect to reflections of the function W_φ, containing n isofermions, as well as isoscalars and isopseudovectors and containing no antiparticles, is expressed by the formula $\Sigma\Gamma_4(m)W_\varphi = 0$, where $m = 1, \dots, n$, which corresponds to conservation of the difference between the number of isofermions and anti-isofermions, i.e. to conservation of hypercharge Y. Consequently, Γ_4 characterizes the hypercharge Y for particles. If Γ_4 and Σ_{12} are diagonal, then for u_1 and u_2 one has $Y = +1$, and for u_3 and u_4 one has $Y = -1$; $T_3 = 1/2$ for u_1, u_3; $T_3 = -1/2$ for u_2, u_4; for antiparticles the signs of T_3 and Y are opposite.

The expression for the hypercharge of quantized fields is

$$\hat{Y}_\varphi = -i \int \left[(\pi\Gamma_4\varphi) - (\pi^+\Gamma_4\varphi^+)\right] dV$$

for a scalar φ,

$$\hat{Y}_\psi = \frac{i}{2} \int [\bar{\psi}, \gamma_4\Gamma_4\psi]\, dV, \qquad \bar{\psi} = \psi^+\gamma_4,$$

for fermions ψ.

3. Let us determine the conditions under which transformations with M^-_{0j} can also characterize an "internal" symmetry. To this end let us consider the transition from the pseudo-Euclidean metric of the group $M^-_{\mu\nu}$ to a Euclidean one; here we shall use Schwinger's results on Euclidean quantum electrodynamics ([3]).

Suppose that Hermitian fields $\psi_1, \dots, \psi_n$ belong to an internal group with a Euclidean metric. Denote the Hermitian operators of the isospin group of rotations in four-dimensional space by T_{ab}; $a, b = 1, 2, 3, 4$,

$$[T_{ab}, T_{cd}] = i[\delta_{ac}T_{bd} - \delta_{ad}T_{bc} - \delta_{bc}T_{ad} + \delta_{bd}T_{ac}]. \tag{3}$$

The matrices $T_{ab} = -T_{ba}$ are antisymmetric. Put, for $j, k = 1, 2, 3$, $M^-_{jk} = T_{jk}$. Since the matrix M^-_{0j} is symmetric, the transition from iM^-_{0j} to T_{4j} also includes a change of symmetry.

For integral isospin T the reflection operator P_T is Hermitian and symmetric; therefore

$$T_{4k} = i\exp\frac{i\pi}{4}P_T \cdot M^-_{2k}\exp\left(-\frac{i\pi}{4}P_T\right) = -P^-_{TM_{0k}}. \tag{4}$$

(The operator M^-_{0k} is anti-Hermitian with respect to states constructed with the aid of the fields ψ.)

In the case of isospin $T = 1/2$, the Hermitian isospinor χ in four-dimensional Euclidean space has 8 components; the Hermitian isospinor u in pseudo-Euclidean space (the "rotation" operators $M^-_{\mu\nu}$) has only 4 components. Therefore the transition to the Euclidean "internal" metric is possible if the isospinor field in the pseudo-Euclidean metric is complex; the additional degree of freedom then corresponds to anti-isofermions.

For the isospinor χ,

$$T_{ab} = \frac{1}{4i}[\alpha_a, \alpha_b],$$

where α_a are symmetric anticommuting 8×8 matrices; the isospin-reflection operator P_T is ···

$P_T = i\alpha_4$. The relation of α_4 to the reflection operator $i\Gamma'_4$ for u (Γ'_4 is Γ_4 in the Majorana representation) is given by the formula $\alpha_4 = \lambda_2\Gamma'_4$, where λ_2 is a two-row antisymmetric Hermitian matrix acting in the space of the two pseudo-Euclidean isospinors u_I and u_{II} and describing the isofermion number.

The transition from M^-_{0j} to T_{4j} is effected by means of the transformation

$$T_{4j} = -i\exp\frac{i\pi}{4}\alpha_4 \cdot M^-_{0j}\exp\left(-\frac{i\pi}{4}\alpha_4\right) = -\alpha_4 M^-_{0j}. \tag{5}$$

Thus, the transition from the three-dimensional isospin group to the four-dimensional isospin group T_{ab} is possible only with conservation of isospin parity, which, according to Sec. 1, is equivalent to conservation of spatial parity.

Representations of the 4-isospin group are labeled by the eigenvalues of the operators $(T^{\pm})^2, T_3^{\pm}$

$$T_3^{\pm} = \frac{1}{2}\left[T_{12} \pm T_{34}\right] \quad \text{and so on.} \tag{6}$$

An essential feature of the group (3) obtained is that four-dimensional symmetry can be realized only in states with hypercharge equal to zero, since the parity P_T anticommutes with T_{j4}:

$$P_{TT_{j4}} + T_{j4}P_T = 0. \tag{7}$$

4. The expression obtained earlier for the electric charge remains valid in the case of isobosons. In the case of isospinor representations, however, there are two possibilities: either the charge Q has another relation to the isospin T, or isofermions do not exist.

The electric charge is determined by such a phase transformation D_Q when (for invariant $W_\varphi\{x, A^1\}$) A^1_μ does not change (being the "3" component of an isopseudovector, or, like T_3, the "12" component of an antisymmetric tensor (1)). One of such transformations may be $\exp(iM_{12}^{-}\alpha) = D_A^0$. In the case of isofermions, however, there is an additional phase transformation $D_Y = 1 + iY\beta$ with real β, where Y is the hypercharge, which commutes with $T_3 = \frac{1}{2}\Sigma_{12}$ and is an isoscalar. The field A^1_μ will be invariant with respect to $D_A^0 \cdot D_Y$, in particular, A^1_μ is invariant with respect to the transformation $\exp i\left(T_3 + \frac{\pi}{2}Y\right)$ or else $P_T \exp i\pi T_3$ for isofermions. In view of this we set, similarly to (2),

$$Q = T_3 + \frac{1}{2}Y. \tag{8}$$

This formula is therefore not only a consequence of relativistic and gauge invariance, but also a consequence of the possibility of passing from the pseudo-Euclidean "internal" space to a Euclidean one.

5. Thus, with conservation of spatial parity P_L, the axiomatic theory contains an additional group of four-dimensional Euclidean rotations T_{ab} with a "spatial" isospinor reflection, which determines the hypercharge.

In the absence of the electromagnetic field (and of vector mesons associated with gauge invariance), the symmetry properties may be "maximal" with respect to what can be given by the axioms of quantum field theory together with the condition of conservation of spatial parity; in this case states can be characterized by the quantum numbers of isospin $T^2 = T_{12}^2 + T_{23}^2 + T_{31}^2, T_3$, and by hypercharge (parity), but not by four-dimensional symmetry in the general case, since $T_{34}P_T = -P_{TT_{34}}$.

Let us compare our results with the conclusions of phenomenological theories. The set of quantum numbers T, T_3, Y coincides with the set in the theory of d' Espagnat and Prentki ([2]). In that theory, however, there is no hidden four-dimensional symmetry obtained by us, which manifests itself when the hypercharge is equal to zero (the quantum numbers $T^{\pm}, T_3^{\pm}$).

The isospin group of the axiomatic theory also does not coincide with the group proposed by Salam and Polkinghorne ([4]), where there is no reflection, the charge is identified with T_{12}, and the three-dimensional isospin with T^{+}.

The relations (3), (7), (8), which determine the symmetry in the axiomatic theory, were postulated in the phenomenological group of A. M. Baldin and A. A. Komar ([5]).

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Note: Figure translations are in progress. See original paper for figures.