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Abstract

Full Text

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ON A CRITERION FOR THE EXISTENCE OF WAVE OPERATORS

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Let H_1, H_2 be self-adjoint operators in a Hilbert space $\mathfrak{H}$; let $P_k,\ k = 1, 2$, be the projector onto the absolutely continuous subspace [1,2] $\mathfrak{H}_k$ of the operator H_k. If the operator $V = H_2 - H_1$ is nuclear*, then, as established by T. Kato [3], the strong limits (wave operators) exist

$$W_\pm(H_2, H_1) = \lim_{t\to\pm\infty} \exp(iH_2t)\exp(-iH_1t)P_1, \tag{1}$$

$$W_\pm(H_1, H_2) = \lim_{t\to\pm\infty} \exp(iH_1t)\exp(-iH_2t)P_2. \tag{2}$$

The operators $W_\pm(H_2, H_1)$ isometrically map $\mathfrak{H}_1$ onto $\mathfrak{H}_2$,

$$W_\pm^*(H_2, H_1) = W_\pm(H_1, H_2),$$

$$W_\pm(H_2, H_1)H_1P_1 = H_2W_\pm(H_2, H_1)P_1,$$

and the indicated properties follow only from the fact that the limits (1), (2) exist. The existence of these limits is of interest for the quantum theory of scattering. However, in applications the conditions of Kato's theorem are usually not fulfilled. Somewhat broader conditions were indicated by S. Kuroda [4], who essentially followed Kato' s method.

Further criteria for the existence of wave operators were obtained in [2,5]. In [2] a new method was proposed, connected with considering functions of operators. Let H_1, H_2 be positive definite, $H_2^{-1} - H_1^{-1} \in \mathfrak{S}_\infty$, and for some integer $n \geqslant 1$, $H_2^{-n} - H_1^{-n} \in \mathfrak{S}_1$. In [2] it was established that under these conditions the limits (1), (2) exist, and moreover

$$W_\pm(H_k, H_l) = W_\mp(H_k^{-n}, H_l^{-n}), \qquad k, l = 1, 2;\ k \neq l. \tag{3}$$

It was also indicated that for $n = 1$ the condition of positive definiteness may be replaced by the condition of bounded invertibility of the operators H_1, H_2.

In ([5]) (devoted mainly to the general theory of S-matrices), the concept of a wave operator was transferred to a pair of unitary operators U_1, U_2:

$$W_{\pm}(U_2, U_1) = \lim_{n \to \pm\infty} U_2^n U_1^{-n} P_1.$$

For this case an analogue of Kato' s theorem was proved in ([5]). It was also shown there that the existence of the limits (1), (2) is ensured by the condition $(H_2 + iI)^{-1} - (H_1 + iI)^{-1} \in \mathfrak{S}_1$. In this case

$$W_{\pm}(H_k, H_l) = W_{\pm}(U_k, U_l), \qquad k, l = 1, 2;\ k \neq l, \tag{4}$$

where $U_k = (H_k - iI)(H_k + iI)^{-1}$.

In connection with the results cited, the question naturally arises of finding a sufficiently general class of functions of operators for which analogues of the relations (3) and (4) are valid. This question is solved in the pres—

* The class of nuclear operators is denoted below by $\mathfrak{S}_1$. The class of all completely continuous operators is denoted by $\mathfrak{S}_\infty$.

the cited paper. At the same time we obtain a new general criterion for the existence of the wave operators (1), (2).

Let Δ be a collection of a finite number of nonintersecting intervals (a_k, b_k), $a_k < b_k \leqslant a_{k+1}$, $k = 1, \dots, m$, lying on the real axis. The cases $a_1 = -\infty$ and $b_m = +\infty$ are not excluded. Let the function $\varphi(z)$ be regular in some open set of the complex plane containing Δ, and at points $\lambda \in \Delta$ let the function $\varphi(\lambda)$ be real and $\varphi'(\lambda) > 0$. Obviously, there exist limits (finite or infinite) $\tilde{a}_k = \varphi(a_k + 0)$, $\tilde{b}_k = \varphi(b_k - 0)$. We shall assume that the intervals $(\tilde{a}_k, \tilde{b}_k)$, $k = 1, \dots, m$, are pairwise nonintersecting. If $\tilde{a}_k = -\infty$ ($\tilde{b}_k = +\infty$), then the corresponding end a_k (b_k) of the interval (a_k, b_k) will be called special for the function $\varphi(z)$. It is clear that there can be no more than one left and one right special end.

A function $\varphi(z)$ with the properties listed above will be called **admissible** for the pair of self-adjoint operators H_1, H_2, if the spectra of the latter are contained in the closure of the set Δ, and the eigenvalues do not coincide with the special ends for the function $\varphi(z)$.

Developing in the proper way the method of the paper [2], one can establish the following assertion.

Theorem. Let the function $\varphi(z)$ be admissible for the pair of self-adjoint operators H_1, H_2, and let $\widetilde{H}_k = \varphi(H_k)$,

$$U_k = (H_k - iI)(\widetilde{H}_k + iI)^{-1}, \qquad \widetilde{U}_k = (\widetilde{H}_k - iI)(\widetilde{H}_k + iI)^{-1},$$

$k = 1, 2$. If

$$(\widetilde{H}_2 + iI)^{-1} - (\widetilde{H}_1 + iI)^{-1} \in \mathfrak{S}_1, \tag{5}$$

then the strong limits $W_\pm(H_k, H_l)$, $W_\pm(U_k, U_l)$ exist, and

$$W_\pm(H_k, H_l) = W_\pm(\widetilde{H}_k, \widetilde{H}_l) = W_\pm(\widetilde{U}_k, \widetilde{U}_l) = W_\pm(U_k, U_l),$$

$$k, l = 1, 2; \quad k \neq l.$$

Let us note that condition (5) is in any case fulfilled if the domains of definition of the operators $\widetilde{H}_1$ and $\widetilde{H}_2$ coincide and $\widetilde{H}_2 - \widetilde{H}_1 \in \mathfrak{S}_1$. We give several examples.

1) Let the operators H_1, H_2 be positive definite and, for some $\alpha < 0$, $H_2^\alpha - H_1^\alpha \in \mathfrak{S}_1$. Then for any $\beta \neq 0$ the operators $W_\pm(H_2^\beta, H_1^\beta)$ exist, and $W_\pm(H_2, H_1) = W_\pm(H_2^\beta, H_1^\beta)$ for $\beta > 0$ and $W_\mp(H_2, H_1) = W_\pm(H_2^\beta, H_1^\beta)$ for $\beta < 0$. This assertion, obviously, strengthens the corresponding result of the paper (2). In particular, the condition $H_2^{-1} - H_1^{-1} \in \mathfrak{S}_\infty$ has turned out to be superfluous.

2) Let the operators H_1, H_2 be positive and $e^{-H_2} - e^{-H_1} \in \mathfrak{S}_1$. Then the strong limits (1), (2) exist, and $W_\mp(H_2, H_1) = W_\mp(e^{-H_2}, e^{-H_1})$. The result presented may be useful in the study of differential operators. This is connected with the fact that the Green' s function of an elliptic boundary-value problem has less smoothness than the Green' s function of the corresponding parabolic problem.

3) Let, for some odd $n > 0$,

$$(H_2^n + iI)^{-1} - (H_1^n + iI)^{-1} \in \mathfrak{S}_1.$$

Then the strong limits $W_\pm(H_2, H_1) = W_\pm(H_2^n, H_1^n)$ exist. For $n = 1$ we again obtain Theorem 2 of the paper (5).

The number of examples could be increased without difficulty.

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Note: Figure translations are in progress. See original paper for figures.