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Abstract

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ON THE PRINCIPLE OF THE CRITICAL POINT IN BANACH SPACES

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1. In this note we consider the problem of finding critical points of a functional $f(x)$, defined in a Banach space B, relative to the manifold

$$N = \{x : \Phi(x) = b_1\}. \tag{1}$$

Here $\Phi(x)$ is an operator mapping B into the Banach space B_1, and $b_1 \in B_1$ is a certain fixed element. We shall further assume that the functional $f(x)$ and the operator $L_\Phi(x)$ are Fréchet differentiable, and denote

$$L_f(x) = \operatorname{grad} f(x), \qquad L_\Phi(x) = \Phi'(x).$$

The operator adjoint to the linear operator $L_\Phi(x)$ will be denoted by $L^*_\Phi(x)$. A point $x_0 \in N$ is called a **conditionally critical point** of the functional $f(x)$ relative to N if there exists an element $h_1 \in B^*_1$ such that

$$L_f(x_0) = L^*_\Phi(x_0) h_1. \tag{2}$$

Many authors have dealt with the finding of critical points relative to manifolds. In the finite-dimensional case the fundamental works here are those of L. A. Lyusternik and L. G. Shnirelman. Some results of this theory were transferred to infinite-dimensional spaces in the works of L. A. Lyusternik, Z. S. Citlanadze, and others ([1–7]). These results include, in particular, the so-called critical-point principle, established in works ([1–7]) for the sphere and certain other manifolds in Hilbert or Banach space. As is known, the critical-point principle makes it possible, under certain assumptions, to establish the existence of a countable number of critical points of an even functional $f(x)$ relative to the sphere.

In the present note we obtain the critical-point principle for manifolds N of general form, given by formula (1).

Let us introduce the following assumptions:

1) The space B is reflexive and N is bounded in norm.

2) $L_f(x)$ is uniformly continuous and bounded in norm on N.
3) $\|L^*_\Phi(x)h_1\| \geq m\|h_1\|$, $m > 0$, $h_1 \in B^*_1$, $x \in N$.
4) The operator $L_\Phi(x)$ satisfies a Lipschitz condition and is bounded on N.
5) There exist such $A, \delta > 0$ that, for $\|x_i - x_0\| \leq \delta$, $x_0 \in N$, there are projectors P_i, $P_i B^* = F_{x_i}$ $(i = 1, 2)$, with

$$\|P_1 - P_2\| \leq A\|x_1 - x_2\|.$$

Here

$$F_x = \{f^* \in B^*,\ f^* = L^*_\Phi(x)h_1,\ h_1 \in B^*_1\}.$$

Remark. If B is a Hilbert space, then condition 5) is a consequence of conditions 3) and 4).

We shall, following ([8]), call an operator (functional) **strongly continuous** if it maps every weakly convergent sequence into a strongly convergent sequence. In what follows, $[A]$ will everywhere denote a certain closed compact homotopic class of sets in N, i.e. a class of compact sets, co-

containing all topological limits of their sets and containing, together with any set, its arbitrary continuous deformations (in N).

Theorem 1. Let assumptions 1)–5) be satisfied. If

$$c = \sup_{A \in [A]} \min_{x \in A} f(x),$$

then there exists a sequence of "almost critical" points $x_n \in N$ such that $\lim_{n\to\infty} f(x_n) = c$ and

$$\lim_{n\to\infty} \left\| L_f(x_n) - L^*_\Phi(x_n)h_1^{(n)} \right\| = 0, \tag{3}$$

where $h_1^{(n)} \in B^*_1$.

The proof uses the method of displacement along the trajectories of a certain differential equation and is based on the following lemma.

Lemma 1. Let assumptions 1)–5) be satisfied. Suppose that for some $x_0 \in N$ and for all $h_1 \in B^*_1$

$$\left\| L_f(x_0) - L^*_\Phi(x_0)h_1 \right\| \geq \alpha > 0. \tag{4}$$

Then, for $x \in \gamma = \{x : \|x - x_0\| \leq \eta\}$, there exists $\xi(x) \in B$ such that:
a) $(L_f(x), \xi(x)) \geq 1$; b) $L_\Phi(x)\xi(x) = 0$; c) $\xi(x)$ satisfies the Lipschitz condition with constant C_ξ and $\|\xi(x)\| \leq K$. Here C_ξ, K, η depend on α, but do not depend on x_0.

In order to obtain a stronger assertion on the existence of a critical point, one must impose, in addition to conditions 1)–5), supplementary conditions on the operators $L_f(x)$ and $L_\Phi(x)$. We shall consider the following assumptions:

2′) The operators $L_f(x)$, $\Phi(x)$, and $L_\Phi(x)$ are strongly continuous.

2″) Relation (3) and $x_n \xrightarrow{\text{sl}} x_0$ imply that

$$\lim_{k\to\infty} \|x_{n_k} - x_0\| = 0.$$

We shall denote the totality of conditions 1), 2″, 3), 4), and 5) by A, and the totality of conditions 1), 2), 2″, 3), 4), and 5) by B.

Theorem 2. Let one of the conditions A or B be satisfied. Then, if

$$c = \sup_{A\in[A]} \min_{x\in A} f(x),$$

there exist $x_0 \in N$ and $h_1 \in B_1^*$ such that $f(x_0) = c$ and

$$L_f(x_0) = L_\Phi^*(x_0)h_1.$$

Remark. Theorem 2 establishes the critical point principle, which is a natural generalization of the principle of minimum (maximum). It is known ([12]; see also [13,14]) that a necessary condition for a minimum of a functional at $x_0 \in N$ is the validity of equality (2). This result is valid under certain assumptions on the regularity of the point $x_0 \in N$. Conditions 3) and 5) are analogues of these regularity conditions.

2. Let us consider some special cases of Theorem 2. Let

$$\Phi(x) = \{\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)\}, \tag{5}$$

where $\varphi_i(x)$ $(i = 0, 1, \dots, n)$ are functionals in B. We impose on the manifold

$$N = \{x : \varphi_i(x) = a_i \ (i = 0, 1, \dots, n)\} \tag{6}$$

and on the functional $f(x)$ the following restrictions:

a) B is reflexive and N is bounded in norm;

b) $L_{\varphi_i}(x)$ $(i = 0, 1, \dots, n)$ satisfy the Lipschitz condition;

c)

$$\left\|\sum_{i=0}^{n} \lambda_i L_{\varphi_i}(x)\right\| \geq m \sum_{i=0}^{n} |\lambda_i| \quad \text{for all } \lambda_i,\ m > 0,\ x \in N;$$

d) $L_f(x)$ is strongly continuous and bounded on N;

e) $L_{\varphi_i}(x)$ are strongly continuous.

Instead of restriction d), one may consider the following restrictions:
d′) let $K \geq 1$ be a fixed constant and let the continuous functionals $\lambda_i(x)$ satisfy the condition

$$\left\|L_f(x) - \sum_{0=i}^{n} \lambda_i(x) L_{\varphi_i}(x)\right\| \leq K \min_{\{a_i\}} \left\|L_f(x) - \sum_{i=0}^{n} a_i L_{\varphi_i}(x)\right\|. \tag{7}$$

Let the set of indices $I = \{0, 1, \dots, n\}$ be divisible into two groups I' and I'' such that for $i \in I'$ the $L_{\varphi_i}(x)$ are strongly continuous, while for $i \in I''$ the $L_{\varphi_i}(x)$ are uniformly continuous and bounded on N, and the operator

$$R(x) = \sum_{i \in I''} \lambda_i(x) L_{\varphi_i}(x)$$

has a continuous inverse operator R^{-1}, $R^{-1}(R(x)) = x$ $(x \in N)$;
d″) in particular, in condition (7) one may take as $\lambda_i(x)$

$$\overline{\lambda}_i(x) = (L_f(x), N_i(x)),$$

where $\{N_i(x)\}$ is a system forming, with $\{L_{\varphi_i}(x)\}_{i=0}^{n}$, a biorthogonal system:

$$(L_{\varphi_i}(x), N_j(x)) = \delta_{ij} \quad (i, j = 0, 1, \dots, n). \tag{8}$$

Let $L_{\varphi_i}(x)$ $(i = 1, \dots, n)$ be strongly continuous, let $\varphi_0(x) = \|x\|$, let $L_{\varphi_0}(x)$ $(x \in N)$ have a continuous inverse operator, and let $(L_f(x), N_0(x)) > 0$ $(x \neq \theta)$, $f(x) > 0$ $(x \neq \theta)$, $f(\theta) = 0$.

Theorem 3. *Let $\Phi(x)$ have the form (5), and suppose that conditions a)–d) are fulfilled (condition d) may be replaced by d′) or d″)). If $c = \sup_{A \in [A]} \min_{x \in A} f(x)$, then there exists $x_0 \in N$ such that $f(x_0) = c$,*

$$L_f(x_0) = \sum_{i=1}^{\infty} \lambda_i L_{\varphi_i}(x_0). \tag{9}$$

Theorem 3 with condition d) follows from Theorem 2 (A); with condition d′) or d″)–from Theorem 2 (B). (Condition b) in this case follows from conditions b) and c).)

Remark 1. For the case $n = 0$ and $f(x) = f(-x)$, stronger results are obtained by considering homotopy classes on the projective sphere (see ([1–4])).

Remark 2. If B is a Hilbert space, then Theorem 3 implies the result of ([5]). This follows from the fact that the condition $|\Delta(x)| \geq c > 0$, where $\Delta(x)$ is the Gram determinant of the system $\{L_{\varphi_i}(x)\}$, is equivalent to condition c). Note that, unlike in ([5]), we do not require that $L_f(x)$ satisfy a Lipschitz condition.

Remark 3. In ([7]) a theorem is formulated whose conclusion coincides with the conclusion of Theorem 3, d′). The proof of this theorem rests on the following lemma:

Lemma. *Suppose* $\{L_{\varphi_i}(x)\}$ $(i = 0, 1, \dots, n)$ *are linearly independent for all* $x \in N$*, satisfy a Lipschitz condition, and* $\|L_{\varphi_i}(x)\| \geq \delta > 0$*. Then there exists a system* $\{N_i(x)\}$ *satisfying the relations (8) and such that, for* $x \in N$*,* $\|N_i(x)\| \leq C$ $(i = 0, 1, \dots, n)$*, and the* $N_i(x)$ *satisfy a Lipschitz condition.*

For such a lemma to be valid, it is necessary that the $L_{\varphi_i}(x)$ be **uniformly** linearly independent in the sense of condition c). Otherwise one can construct examples in which $\sup_{x \in N} \|N_i(x)\| = \infty$.

Let now B be a Hilbert space and

$$\Phi(x) = \{\varphi_i(x)\}_{i=1}^{\infty},$$

$$N = \{x : \varphi_i(x) = a_i \ (i = 1, 2, \dots)\}. \tag{10}$$

Assumptions:

a[0]) N is bounded in norm;

b[0]) $L_f(x)$, $L_{\varphi_i}(x)$ are strongly continuous;

c[0])

$$m \left(\sum_{i=1}^{k} \lambda_i^2 \right)^{1/2} \leq \left\| \sum_{i=1}^{k} \lambda_i L_{\varphi_i}(x) \right\| \leq M \left(\sum_{i=1}^{k} \lambda_i^2 \right)^{1/2} \quad (x \in N)$$

for all λ_i and $k = 1, 2, \dots;\ 0 < m \leq M$;

d[0]) the operators $L_{\varphi_i}(x)$ satisfy the Lipschitz condition with constants C_i, and

$$\sum_{i=1}^{\infty} C_i^2 < \infty.$$

Theorem 4. Let N have the form (10), and suppose that assumptions a[0])–d[0]) are satisfied. Then, if

$$C = \sup_{A \in [A]} \min_{x \in A} f(x),$$

there exist $x_0 \in N$ and a sequence $\{\lambda_i\} \in l^2$ such that $f(x_0) = C$ and

$$L_f(x_0) = \sum_{i=1}^{\infty} \lambda_i L_{\varphi_i}(x_0).$$

Remark 1. Instead of condition b^0), one may consider conditions b^{00}, b^{000}, analogous to conditions d$'$, d$''$ of Theorem 3.

Remark 2. Condition c^0) means that the system $\{L_{\varphi_i}(x)\}$ is a Riesz basis (see (9)) in its closed linear hull G_x (for every $x \in N$).

Remark 3. The theorem of E. S. Tsitlanadze [6] follows from Theorem 4 as a special case. Indeed, one may note that the conditions imposed in that paper on the Gram determinant of the system $\{L_{\varphi_i}(x)\}$ imply that this system is a Bari basis in G_x (see (10)). Then it is also a Riesz basis in G_x, and the relations c^0) are satisfied.

We indicate one more special case of Theorem 2.

Let $\Phi(x)$ be an operator acting from the space $L^2_{\sigma_1}(a_1, b_1)$ into $L^2_{\sigma_2}(a_2, b_2)$, and let $N = \{x : \Phi(x) = y\}$, where y is a fixed element.

Suppose that the linear operator $L^*_\Phi(x)$, for every $x \in N$, generates the Riesz-Fischer kernel $\widetilde{K}(\xi, u; x)$, i.e., for every $\xi \in (a_1, b_1)$

$$\widetilde{K}(\xi, u; x) = A(x) K(\xi, u),$$

where $A(x)$ is a linear operator (for every $x \in N$), and $K(\xi, u)$ is the kernel (in Bochner' s sense) of some unitary operator.

Suppose the following assumptions are satisfied:

α) N is bounded in norm;

β) $0 < m\|h\| \le \|A(x)h\| \le M\|h\|$ $(x \in N)$; $h \in L^2_{\sigma_1}(a_1, b_1)$;

γ) $L_f(x)$, $L_\Phi(x)$, $\Phi(x)$ are strongly continuous;

δ) $L_\Phi(x)$ satisfies the Lipschitz condition.

Theorem 5. Suppose assumptions α)–δ) are satisfied, and

$$C = \sup_{A \in [A]} \min_{x \in A} f(x).$$

Then there exist $x_0 \in N$ and a function $h(u) \in L^2_{\sigma_2}(a_2, b_2)$ such that $f(x_0) = C$,

$$L_f(x_0) = \frac{d}{d\xi} \left[\int_{a_2}^{b_2} \widetilde{K}(\xi, u; x_0)\, h(u)\, d\sigma_2(u) \right].$$

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Note: Figure translations are in progress. See original paper for figures.