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Abstract

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MATHEMATICS

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ON LINEAR OPERATORS IN ORLICZ COORDINATE SPACES

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Let $P(u)$ and $Q(u)$ $(-\infty < u < \infty)$ be continuous, even functions, monotonically increasing for $u \geq 0$, with $P(u) = Q(u) = 0$. Let $\{a_{ik}\}_1^\infty$ be an infinite matrix satisfying the conditions

$$\sum_{k=1}^{\infty} P(a_{ik}) \leq C_1, \qquad \sum_{i=1}^{\infty} Q(a_{ik}) \leq C_2. \tag{1}$$

We study the conditions for continuity and complete continuity of the linear matrix operator

$$y = Ax, \tag{2}$$

where $x = (\xi_1, \dots, \xi_k, \dots)$, $y = (\eta_1, \dots, \eta_i, \dots)$, and

$$\eta_i = \sum_{k=1}^{\infty} a_{ik}\xi_k, \tag{3}$$

as an operator acting from the Orlicz coordinate space l_M ([1]) into some Orlicz coordinate space l_{M_2}.

A number of conditions for continuity and complete continuity of the operator under consideration were obtained by Yu. I. Gribanov ([2,3]) and others. Here new conditions are formulated.

§ 1. We first give several auxiliary assertions.

Lemma. *In order that the N-function $M_1(u)$ satisfy the Δ_2-condition for small u (for all u), it is necessary and sufficient that there exist an N-function $M_2(v)$ such that*

$$M_1(auv) \geq M_1(u)M_2(v) \tag{4}$$

for small u (for all u) and small v.

The element $z = (\xi_1\eta_1, \dots, \xi_k\eta_k, \dots)$ will be called the $\odot$-product of the elements $x = (\xi_1, \dots, \xi_k, \dots)$ and $y = (\eta_1, \dots, \eta_k, \dots)$, and will be denoted by $x \odot y$.

Theorem 1. *In order that the $\odot$-product $x \odot y$ belong to the space l_{M_1} for every pair of elements $x \in l_{M_2}$ and $y \in l_{M_3}$, it is necessary and sufficient that the N-functions $M_1(u)$, $M_2(u)$, and $M_3(u)$ satisfy the relation*

$$M_1(auv) \leq M_2(u) + M_3(v) \tag{5}$$

for some $a > 0$ and all $u, v \leq u_0$.

An analogous theorem for Orlicz function spaces was proved by T. Ando ([4]). The assertions of these theorems are valid for any finite number of factors.

Let $\mathfrak{M}$ be the class of coordinate Banach spaces E possessing the following properties:

1. From the condition $\lim_{n\to\infty} \|x_n\| = 0$ it follows that x_n converges to zero coordinatewise.
2. From the condition $\lim_{n\to\infty} \|x_n\| = 0$ there follows the existence in E of such an ele-

element x_0 and such a subsequence x_{n_p} $(p = 1, 2, \dots)$ that $|\xi_k^{(n_p)}| \leq |\xi_k^0|$ for all p and k ($\xi_k^{(n)}$ is the k-th coordinate of the element x_n).

3. From the fact that the sequence $x_n \in E$ converges coordinatewise to x, it follows that
$$\|x\| \leq \lim_{n\to\infty} \|x_n\|.$$

Theorem 2. Let E and E_1 be spaces of the class $\mathfrak{M}$. Let the matrix $\{a_{ik}\}_1^\infty$ define the operator (2)–(3) acting from E into E_1. Then the operator (2)–(3) is continuous.

This theorem is an analogue of a theorem of Banach ([5], pp. 74-75) for spaces of measurable functions.

It is easy to see that Orlicz coordinate spaces belong to the class $\mathfrak{M}$.

§ 2. We shall say that the function $\Phi_1(u)$ is less than the function $\Phi_2(u)$ (and write $\Phi_1(u) < \Phi_2(u)$) if $\Phi_1(u) \leq \Phi_2(\omega u)$ for $u \leq u_0$. We shall say that $\Phi_1(u)$ is essentially less than $\Phi_2(u)$ (and write $\Phi_1(u) \ll \Phi_2(u)$) if

$$\lim_{u\to\infty} \frac{\Phi_1(u)}{\Phi_2(\varepsilon u)} = 0$$

for every $\varepsilon > 0$. The functions $\Phi_1(u)$ and $\Phi_2(u)$ are called equivalent ($\Phi_1(u) \sim \Phi_2(u)$) if simultaneously $\Phi_1(u) < \Phi_2(u)$ and $\Phi_2(u) < \Phi_1(u)$.

A simple consequence of Hölder' s inequality for Orlicz coordinate spaces is the following.

Theorem 3. If the second of conditions (1) is satisfied, then the operator (2)–(3) acts continuously from the space l_1 of summable sequences into any space l_{M_1} for which $M_1(u) < Q(u)$.

Theorem 4. If the series

$$\sum_{i=1}^{\infty} Q(\lambda a_{ik})$$

converge uniformly with respect to k for every $\lambda > 1$, then under the hypotheses of Theorem 3 the operator A acts from l_1 into h_{M_1} and is completely continuous.

Here h_{M_1} denotes the closure in l_{M_1} of the set of elements with a finite number of coordinates different from zero.

Everywhere below it is assumed that the monotone functions under consideration possess the following property: from the condition $\Phi_1(u) \ll \Phi_2(u)$ it follows that the functions $\Phi_1(u)/\Phi_2(u)$ and $\Phi_2^{-1}(v)/\Phi_1^{-1}(v)$, tending to zero as $u, v \to 0$, are monotone for small values of the arguments. By $f^{-1}(v)$ is denoted the function inverse to the function $f(u)$.

If $Q(u) \ll u$ and $u < Q(u)$, then the class of spaces l_{M_1} satisfying the condition of Theorem 3 is nonempty. In the first case $Q(u)$ is equivalent to some N-function; in the second case the relation $M_1(u) < Q(u)$ is valid for any N-function $M_1(u)$.

If conditions (1) guarantee the continuity of the operator A as an operator acting from the space l_M into some coordinate space contained in the space m of bounded sequences, then the following relation is satisfied: $N(v) < P(v)$, where $N(v)$ is the function complementary to $M(u)$. From Young' s inequality it follows:

Theorem 5. Let the first of conditions (1) be satisfied and let $N(v) \sim P(v)$. Then the operator (2)–(3) acts continuously from the space l_M into the space m.

Below it is assumed that $N(v) \ll P(v)$. Then the function $R^{-1}(v) = v/N^{-1}[P(v)]$ is monotone for small v, and its inverse function $R(u)$ is equivalent to some N-function.

Theorem 6. Let $N(v) \ll P(v)$. Let the function $Q[R(u)]$ be equivalent to some N-function $M_1(u)$ satisfying, for small u, the Δ_2-condition and the Δ_1-condition:

$$M_1(\alpha uv) \leq M_1(u) M_1(v) \quad (u, v \leq u_0).$$

If $M_1(u) \sim M(u)$ with $M_1(u) \ll M(u)$, then the operator (2)–(3) acts continuously from the space l_M into any space l_{M_2} for which $M_2(v)$ satisfies inequality (4).

If the function $N_1(v)$, complementary to $M_1(u)$, also satisfies the Δ_1-condition for all u, v, then the operator A acts continuously from l_M into l_{M_1}.

If the functions $P(u)$ and $Q(u)$ are equivalent and, then the function $Q[R(u)]$ is equivalent to the N-function $M(u)$.

Theorem 7. If the series $\sum_{i=1}^{\infty} Q(a_{ik})$ converge uniformly with respect to k, then under the conditions of Theorem 6 the operator A acts from l_M into h_{M_2} (or into h_M) and is completely continuous.

It follows from Theorems 3, 5 and 6, in the particular case (without estimating the norm of the operator), the following theorem of Hardy, Littlewood and Pólya ([6], p. 239):

Let $r, t > 0$,

$$\sum_{k=1}^{\infty} |a_{ik}|^r \le C_1, \qquad \sum_{i=1}^{\infty} |a_{ik}|^t \le C_2. \tag{6}$$

Let $p \ge 1$ and $p' = \dfrac{p}{p-1} \ge r$. Let

$$t \ge \left(1 - \frac{r}{p'}\right) p. \tag{7}$$

Then the operator (2)–(3) acts continuously from l_p into l_{p_1}, where

$$p_1 = \frac{t}{1 - r/p'} \quad (l_\infty = m, \text{ if } p' = r).$$

This theorem is naturally supplemented by the following assertion:

Theorem 8. If $p' > r$ and the series $\sum_{i=1}^{\infty} |a_{ik}|^t$ converge uniformly with respect to k, then under the conditions of the preceding theorem the operator A is completely continuous as an operator from l_p into l_{p_1}.

The conditions (6) are less restrictive than the usual sufficient conditions for complete continuity of matrix operators in the spaces l_p $(p > 1)$ (see, for example, ([7], pp. 322–323). Thus, for example, from the well-known Koch condition

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |a_{ik}|^2 < \infty$$

one cannot obtain an assertion on complete continuity in l_2 of the diagonal operator

$$A = \begin{vmatrix} \alpha_1 & 0 & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & \alpha_n & . & . \\ . & . & . & . & . & . & . \end{vmatrix},$$

where $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$. At the same time, the complete continuity of such an operator follows from Theorem 8.

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Note: Figure translations are in progress. See original paper for figures.