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Abstract

Full Text

MATHEMATICS

P. P. BELINSKII

ON THE CONTINUITY OF SPATIAL QUASICONFORMAL MAPPINGS AND ON LIOUVILLE' S THEOREM

(Presented by Academician M. A. Lavrent' ev on June 22, 1962)

As is known, there is an essential difference between plane and spatial conformal mappings, consisting in the fact that the set of conformal mappings of space (the case $n > 2$) includes only inversions with respect to spheres. This last assertion belongs to Liouville and was proved by him for mapping functions differentiable a sufficient number of times.

In connection with Liouville' s theorem, M. A. Lavrent' ev repeatedly expressed the hypothesis that a quasiconformal mapping which is, locally, close to a conformal one must, on the whole, be close to it and, consequently, under an appropriate normalization, close to the identity mapping; however, he succeeded in proving this only in a number of special cases ⁽¹⁾. In addition, M. A. Lavrent' ev weakened the differentiability requirements in the proof of Liouville' s theorem.

Yu. G. Reshetnyak ⁽²⁾ proved Liouville' s theorem under the following assumptions: every n -dimensional topological mapping that carries an infinitely small ball into a ball is a superposition of a finite number of inversions with respect to spheres. In addition, he proved that if a mapping is $(1 + \varepsilon)$ -quasiconformal (carries an infinitely small ball into an ellipsoid with ratio of the largest semi-axis to the smallest $a/b \leq 1 + \varepsilon$), then as $\varepsilon \rightarrow 0$ the mapping tends, inside the domain, to a conformal mapping.

However, the question of the closeness of mappings in a closed domain, i.e., in essence, M. A. Lavrent' ev' s hypothesis, remained open. In connection with this, one further question arose: whether the normalized functions realizing quasiconformal mappings of the n -dimensional ball form a family of uniformly bounded and equicontinuous functions. Here normalization means the fixing of such a number of interior or boundary points as ensures uniqueness in the conformal mapping. The following theorem gives an answer to these questions.

Theorem. The family $\{Y = f(X)\}$, $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n)$, $n > 2$, of q -quasiconformal mappings of the n -dimensional unit ball, normalized in the above sense, has the following properties:

1. There exists a constant q_0 , depending on the normalization and on the di-

mension of the space, such that for $q < q_0$ the family is uniformly bounded and equicontinuous.

2. $\rho(f(X), X) \leq \lambda(\varepsilon)$, where $\varepsilon = q - 1$, ρ is the distance in n -dimensional space and $\lambda(\varepsilon)$ is a function depending only on the normalization and such that

$$\lim_{\varepsilon \rightarrow 0} \lambda(\varepsilon) = 0.$$

The proof is based on a lemma on equicontinuity.

Lemma. The family of q -quasiconformal mappings of the n -dimensional ball which leave the center fixed and one point on the boundary fixed is uniformly bounded and equicontinuous in every closed subdomain of the ball.

The proof of this lemma is based on previously known assertions of a similar type, in which the character of equicontinuity depended on

the volume of the domain (see, for example, (3)), and the theory of extremal lengths, developed recently for space by B. V. Shabat (4,5) and Gehring (6). This lemma makes it possible to connect the question of convergence of a sequence of mappings inside a domain with the question of convergence of the domains to a kernel, and to develop a theory similar to Carathéodory's theory for conformal mappings. Hence it follows that, for sufficiently small $q - 1$, the family of q -quasiconformal mappings is uniformly bounded and equicontinuous.

Let us illustrate the idea of the proof in the following particular, but sufficiently characteristic, case. We shall show that, for sufficiently small $q - 1$, a ball cannot be mapped onto a domain containing a sufficiently long and thin cylinder (the cylinder belongs to the domain, but its boundary does not). Assuming the contrary, we obtain a sequence of mappings which, by means of auxiliary transformations, can be converted into a sequence of mappings of the ball onto domains whose kernel will be the cylinder. This, however, contradicts what was said above and Liouville's theorem. The same reasoning shows that, for sufficiently small $\varepsilon = q - 1$, the q -quasiconformal image of a ball must have a boundary sufficiently close to a sphere, so to speak, in the tangent sense.

In conclusion we note that $q_0 \neq \infty$, i.e., for sufficiently large q the family of q -quasiconformal mappings of a ball is not uniformly bounded under any normalization. This follows from examples whose basis is the following mapping of a ball onto an infinite cylinder. A disk is mapped conformally onto a strip, with axial symmetry preserved. By rotation, from this mapping one can obtain a mapping of a ball onto a cylinder which, as a simple calculation shows, turns out to be q -quasiconformal with some q that is easily computed.

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