MATHEMATICS

R. P. BOAS, K. I. RAHMAN

1962

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Abstract

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R. P. BOAS, K. I. RAHMAN

SOME INEQUALITIES FOR POLYNOMIALS AND ENTIRE FUNCTIONS

(Presented by Academician M. A. Lavrentiev on 11 VIII 1962)

I. I. Ibragimov and R. G. Mamedov ([1]) proved an inequality for the norm in the space L^p of a polynomial that does not vanish inside the unit circle. The following theorem gives, possibly, the best inequality of this kind.

Theorem 1. If $P_n(z)$ is a polynomial of degree $\leq n$, $P_n(z) \neq 0$ for $|z| < 1$, then for $R > 1$ and $1 \leq p \leq \infty$ the inequality

$$\|P_n(Re^{i\theta})\|_p \leq \frac{\|1 + R^n e^{in\theta}\|_p}{\|1 + e^{in\theta}\|_p} \|P_n(e^{i\theta})\|_p \tag{1}$$

holds.

Inequality (1) can also be written in the form

$$\|P_n(Re^{i\theta})\|_p \leq \frac{p+1}{2^{p+1}} \mathrm{B}\left(\frac{1}{2}, \frac{1}{2}p + 1\right)^{1/p} (R^n - 1)\times$$

$$\times \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left[1 + 4\cos^2\omega\, (R^{n/2} - R^{-n/2})^{-2}\right]^{p/2} d\omega \right\}^{1/p} \|P_n(e^{i\theta})\|_p^*,$$

where B is the beta function. In the case $p = \infty$ this inequality was proved by Ankeny and Rivlin ([2]). The proof is based on Boas' s interpolation formula ([3]) for entire functions and on the theorem of B. Ya. Levin ([4]), p. 412.

The following theorem is also valid for entire functions of finite degree.

Theorem 2. If $f(z)$ is an entire function of degree not exceeding τ, and

$$|f(x + iy)| \leq |f(x - iy)|, \qquad y > 0,$$

then for $y > 0$ the estimate

$$\int_{-\infty}^{\infty} |f(x + iy)|^p\, dx \leq \frac{\displaystyle\int_0^{2\pi/\tau} |\cos\tau(x + iy)|^p\, dx}{\displaystyle\int_0^{2\pi/\tau} |\cos\tau x|^p\, dx} \int_{-\infty}^{\infty} |f(x)|^p\, dx. \tag{2}$$

holds.

Inequality (2) can also be written in the form

$$\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \le \int_{-\infty}^{\infty} |f(x)|^p \, dx \, \frac{\int_0^{2\pi} (1-\sin^2\omega \operatorname{sech}^2 s)^{p/2} \, d\omega}{2\mathrm{B}\left(\frac{1}{2p}+\frac{1}{2}, \frac{1}{2}\right)} \operatorname{ch} \tau y. \quad (3)$$

* As is shown by the example of the function $P_n(z) = 1 + z^n$.

In the case $p = \infty$, inequality (3) takes the form

$$|f(x+iy)| \le (\operatorname{ch} \tau y) \sup_{-\infty<x<\infty} |f(x)|$$

(([5], [3])),

and in the case $p = 2$ (3)

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx \le (\operatorname{ch} 2\tau y) \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

Northwestern University
Evanston, Illinois, USA

Regional Engineering College
Srinagar, Kashmir, India

Received
7 VIII 1962

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Note: Figure translations are in progress. See original paper for figures.