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Abstract

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MATHEMATICS

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ON THE TOPOLOGICAL STRUCTURE OF FACTOR SPACES OF LOCALLY BICOMPACT GROUPS

(Presented by Academician P. S. Aleksandrov, 24 III 1962)

Let G be a locally bicompact group, H a closed subgroup of G, and $B = G/H$. By a well-known theorem of Yamabe, there is in G an open subgroup Γ that is projective-Lie (see ([5],[20],[21])). Therefore the space B decomposes into a sum of pairwise disjoint open-closed sets, each of which is homeomorphic to a factor space of the group Γ. For the study of factor spaces of projective-Lie groups one may use the following representation as an inverse limit, analogous to the decomposition, proposed by L. S. Pontryagin, of a group into a series of Lie groups ([14]).

Theorem 1. *Let G be a projective-Lie group, H a closed subgroup of G, and $B = G/H$. In G there is a decreasing transfinite sequence of bicompact normal divisors Y_α, $\alpha \leqslant \theta$, such that, if $B_\alpha = G/HY_\alpha$ and φ^α_β, $\beta < \alpha$, is the natural projection of B_α onto B_β, then the following conditions hold: 1) φ^α_β are open proper mappings; 2) B_1 is a manifold, $B_\theta = B$; 3) for every $\alpha < \theta$ the mapping $\varphi^{\alpha+1}_\alpha$ is a locally trivial fibration with a compact manifold as fiber; 4) for every limit transfinite number $\gamma \leqslant \theta$*

$$B_\gamma = \lim_{\leftarrow}\{B_\alpha, \varphi^\alpha_\beta;\ \beta < \alpha < \gamma\}.$$

Finally, if the space B is finite-dimensional, then the sequence Y_α can be chosen so that all mappings $\varphi^{\alpha+1}_\alpha$ are finite-sheeted coverings.

This theorem is essentially proved in ([16]). The proof is based on the results of Gleason ([4]) and Iwasawa ([8]). The last assertion of Theorem 1 can also be formulated in the following way:

Theorem 1 . *Let G be a projective-Lie group, H its closed subgroup, and $B = G/H$. If the factor space B is finite-dimensional, then in G there is a*

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bicompact normal divisor Y such that G/Y is a Lie group and the set YH/H is zero-dimensional.

From Theorem 1 the following proposition follows.

Theorem 2. *A finite-dimensional factor space of a locally bicompact group admits a locally trivial fibration with base a manifold and fiber homeomorphic to the generalized Cantor discontinuum D^{τ}.*

With the aid of Theorems 1 and 2, the following known assertions are easily obtained; in the book of Montgomery and Zippin ([10]) they are proved for groups satisfying the first axiom of countability.

Theorem 3. *Let B be a finite-dimensional locally connected factor space of a projective-Lie group G by a closed subgroup H. Then in G there is a bicompact normal divisor Y such that $Y \subset H$ and G/Y is a Lie group; thus, B turns out to be a factor space of the Lie group G/Y.*

Corollary. If G is an arbitrary locally bicompact group and the quotient space $B = G/H$ is finite-dimensional and locally connected, then B is a manifold.

Theorem 4. Let G be a locally bicompact group and H a closed subgroup in G. In order that the quotient space G/H be a manifold, it is necessary and sufficient that all small subgroups of the group G be contained in H.

Theorem 5. Let G be a projective-Lie group and H such a closed subgroup in G that the quotient space G/H is finite-dimensional. Then in G there is a bicompact normal divisor Y such that $Y \subset H$ and the group G/Y is finite-dimensional.

In other words, if the group G acts effectively on its finite-dimensional quotient space, then G is finite-dimensional.

From Theorems 1 and 2 there also follow the following generalizations of results of L. N. Ivanovskii ([7]) and V. I. Kuz' minov ([9]) on the dyadicity of bicompact quotient spaces and the structure of zero-dimensional bicompact groups.

Theorem 6. A zero-dimensional quotient space of a locally bicompact group is homeomorphic to a direct product of a discrete set by D^{τ}.

Theorem 7. A quotient space of a locally bicompact group is the image of the direct product of a discrete set by D^{τ} under a proper mapping. If, in addition, the dimension of the quotient space is $\leq n$, then the corresponding mapping can be chosen to be $(n+1)$-to-one.

The results of Hartman and Hulanicki on the cardinality of locally bicompact groups and their dense subsets extend to quotient spaces of locally bicompact groups (see ([17], [18])).

With the aid of Theorem 2 one can give a new proof of the theorem of B. A. Pasynkov ([12]) on the coincidence of different definitions of dimension for

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quotient spaces of locally bicompact groups. Another application to dimension theory is the following

Theorem 8. Every n-dimensional closed subset of a finite-dimensional quotient space of a locally bicompact group is perfectly n-dimensional in the sense of V. I. Ponomarev ([1], [13]).

By virtue of the results of V. I. Ponomarev this theorem follows from the following proposition:

Theorem 8 . Every n-dimensional closed subset of a finite-dimensional quotient space of a locally bicompact group is an $(n+1)$-fold image of a discrete sum of zero-dimensional bicompacta under a closed mapping.

The proof of Theorem 8′ is based on the following lemma.

Lemma. Let Φ be a closed subset in a finite-dimensional quotient space B of a locally bicompact group G. There exists a locally trivial decomposition $p : B \to \overline{B}$ with base $\overline{B}$, decomposing into a discrete sum of spaces of countable weight, and with zero-dimensional bicompact fiber, such that $\dim p\Phi = \dim \Phi$.

With the aid of this lemma and Theorem 2 one can give a new proof of the theorem of A. V. Zarelua ([6]) on the coincidence of different definitions of dimension for closed subsets of finite-dimensional quotient spaces of locally bicompact groups (this also follows directly from Theorem 8, since, as was proved by V. I. Ponomarev ([1]), for perfectly n-dimensional spaces all dimensions coincide).

The next circle of applications of Theorems 1 and 2 is connected with the study of decompositions of locally bicompact groups and their quotient spaces induced by closed subgroups. For example, one obtains a simple proof of Mostert' s theorem ([11]) on the local triviality of decompositions over a finite-dimensional base, as well as the following two theorems.

Theorem 9. Let G be a projective-Lie group, and let H and F be closed subgroups in G, with $H \supset F$ and H/F a manifold. Then the fibration of the space G/F over G/H is locally trivial.

As is known, not every fibration of a locally bicompact group by a closed subgroup is locally trivial. However, such a fibration always satisfies the covering homotopy theorem for polyhedra (([2]); ([15]), p. 32). The latter assertion can be strengthened somewhat.

Theorem 10. Let G be a locally bicompact group, and let H and F be closed subgroups in G, with $H \supset F$. The fibration of the space G/F over G/H satisfies the covering homotopy theorem (for arbitrary spaces).

As is known, under a mapping of a locally bicompact group onto its quotient space, the component of the identity is not necessarily mapped onto the whole connected component of the identity class of the quotient space (see, for example, ([10])). The following lemma shows that the mapping is onto a dense subset.

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Lemma. Let $B = G/H$ be the quotient space of a locally bicompact group G by a closed subgroup H; let G_0 be the component of the identity in G; let B_0 be the component of the identity class in B; and let p be the canonical mapping of G onto B. Then $p^{-1}B_0 = [G_0 H]$.

From this lemma and Mostert' s theorem it follows that

Theorem 11. Every quotient space of a locally bicompact group is homeomorphic to the topological product $N \times D^\tau \times B_0$, where N is a discrete set, D^τ is a generalized Cantor discontinuum, and B_0 is a connected quotient space of some projective-Lie group.

This theorem shows that the study of the topological structure of connected quotient spaces is of particular interest.

Lemma. Let B be a connected quotient space of a locally bicompact group G by a closed subgroup; let L_B be the component of linear connectedness of the identity class e_B in B (i.e., the set of all points in B that can be joined to e_B by paths). The set L_B is dense in B.

With the help of this lemma, for connected finite-dimensional quotient spaces one obtains the following theorem.

Theorem 12. Every connected finite-dimensional quotient space of a locally bicompact group has weight not exceeding the cardinality of the continuum. If this quotient space is bicompact, then it has countable weight.

The countable character of a connected finite-dimensional bicompact group was proved in A. Weil' s book ([3]). For locally bicompact groups, an analogous assertion follows from the fact that, by the Iwasawa–Mal' cev theorem, every connected locally bicompact group is the topological product of a connected bicompact group and Euclidean space. As for quotient spaces, here the assumption of bicompactness in the second assertion of Theorem 12 is essential, as the following example shows.

Example. Let G_0 be a connected Lie group containing a discrete subgroup X, which is a free group with a countable set of generators $a_1, a_2, \ldots$. Let Y be an arbitrary zero-dimensional bicompact group whose space is homeomorphic to the generalized Cantor discontinuum D^c, where c is the cardinality of the continuum. In D^c there is a countable dense subset $b_1, b_2, \ldots$ (see, for example, ([19])). Denote by Y_1 the subgroup in Y generated (in the algebraic sense) by the elements $b_1, b_2, \ldots$. Let f be a homomorphism of the group X onto Y_1 that sends the elements a_i to the corresponding elements b_i. Consider the group $G = G_0 \times Y$ and in it the closed subgroup H formed by elements of the form (x, fx), $x \in X$. Let $B = G/H$. The quotient space B is connected, finite-dimensional, and has continuum weight.

Remark. Let B_1 be a manifold over which, by virtue of Theorem 2, a connected finite-dimensional quotient space B is fibered. It can be shown that if the

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fundamental group of the manifold B_1 has a finite number of generators, then B has countable weight.

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