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Abstract

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ON ANALYTIC SOLUTIONS OF ESSENTIALLY NONLINEAR EQUATIONS

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By an essentially nonlinear equation we mean an equation of the form $\Omega(y, \lambda) = 0$, if the operator $\Omega'_y(y, 0)$ has no inverse for any y. To equations of this kind may be reduced bifurcation cases, the resonance case, and a number of other critical cases in the theory of nonlinear equations. In the present paper we study the case when essentially nonlinear equations have analytic solutions, determine their number, and construct majorant series. Two results obtained by applying the theorems of the paper to concrete problems are given as examples.

Theorem 1. *Let:*

1. *X, Y, Z, U be complex B-spaces, with $Y = Z \cap U$, $Z = X \dotplus Y$.*
2. *The linear operator A has the following properties: a) $A(Z) = U$; b) $A(X) = 0$; c) there exists a bounded inverse A^{-1}, with $A^{-1}(U) = Y$.*
3. *The operator B is defined on the plane (Z, λ) (λ is a complex parameter); it is analytic in z and λ (1) in a neighborhood of each point of the form $(z, 0)$; $B(Z, \lambda) \subset X \dotplus U$.*
4. *The operator α projects $X \dotplus U$ onto X, i.e., if $x \in X$ and $u \in U$, then $\alpha(x + u) = x$.*
5. *$\widetilde{X}$ is the set of all simple solutions belonging to X of the equation*

$$\alpha B(z, 0) = 0, \tag{1}$$

i.e., $\tilde{x} \in \widetilde{X}$ if and only if: a) $\tilde{x} \in \widetilde{X}$; b) $\alpha B(\tilde{x}, 0) = 0$; c) the operator $\alpha B'_z(x, 0)$ has a bounded inverse from X to X.

Then:

1. *If $\tilde{x} \in \widetilde{X}$, then there exists a unique solution, continuous in some neighborhood of the point $\lambda = 0$, of the equation*

$$A(z) = \lambda B(z, \lambda), \tag{2}$$

satisfying the condition $z(0) = \tilde{x}$. *For sufficiently small* $|\lambda|$, *this solution can be represented in the form of a convergent power series*

$$z(\lambda) = \tilde{x} + \sum_{i=1}^{\infty} z_i \lambda^i. \tag{3}$$

2. *If equation (1) has no multiple (non-simple) solutions in* X, *then equation (2) has no other solutions, defined and continuous in some neighborhood of the point* $\lambda = 0$, *apart from those indicated in 1.*

Let us note that the first assertion of part 1 of this theorem may also be formulated as follows:

If $\tilde{x} \in \widetilde{X}$, *then there exist positive numbers* $\Lambda(\tilde{x})$ *and* $M(\tilde{x})$ *such that for* $|\lambda| \leq \Lambda(\tilde{x})$ *equation (2) has a unique solution* $z(\lambda)$ *satisfying the inequality* $\|z(\lambda) - \tilde{x}\| \leq M(\tilde{x})$.

Let the series $w(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i$, where $a_i \geq 0$, have a nonzero radius of convergeness. If the series $z(\lambda) = \sum_{i=0}^{\infty} z_i \lambda^i$ is majorized by the series $w(\lambda)$, i.e., if the inequalities $\|z_i\| \leqslant a_i$ hold, we shall write $z(\lambda) \preccurlyeq w(\lambda)$.

Theorem 2. Let the function

$$F(w, \lambda) = \sum_{i,j=0}^{\infty} \alpha_{ij} w^i \lambda^j \quad (\alpha_{ij} \geqslant 0)$$

be such that from $z(\lambda) \preccurlyeq w(\lambda)$ it follows that

$$B(\tilde{x} + z, \lambda) \preccurlyeq F(w, \lambda).$$

Then the series (3) is majorized by the series $\|x\| + w(\lambda)$, where $w(\lambda)$ is the unique analytic root of the equation

$$w = \frac{a}{1 + aF'_w(0,0)} [F(w, \lambda) - F(0,0)] + \lambda b F(w, \lambda),$$

satisfying the condition $w(0) = 0$.

Here $a = \|[aB'_{\tilde{x}}(\tilde{x}, 0)]^{-1}\|$, $b = \|A^{-1}\|$.

Example 1. Let V be a vector; $f(V, t)$ a vector-analytic function of V, the coefficients of whose series are 2π-periodic functions of t; $\tilde{f}(V)$ is obtained from $f(V, t)$ by replacing all coefficients by their mean values. Suppose further that the equation $\tilde{f}(V) = 0$ has exactly n solutions $V_1, V_2, \ldots, V_n$, and that all of them are simple, i.e.,

$$\det J(\tilde{f}, V_i) = \det \left(\frac{\partial f}{\partial V} \right)_{V_i} \neq 0.$$

Then:

1. The differential equation

$$\dot{V} = \lambda f(V, t) \tag{4}$$

for sufficiently small $|\lambda|$ also has exactly n 2π-periodic solutions $V_1(\lambda), V_2(\lambda), \dots, V_n(\lambda)$, continuous in λ; they expand in series in powers of λ. If the equation $\tilde{f}(V) = 0$ has no solutions, then equation (4) has no 2π-periodic solutions continuous in λ at the point 0.

2. R_i, the radius of convergence of the series $V_i(\lambda)$, is estimated by the formula

$$R_i \geqslant \max_{w>0} \frac{w - \dfrac{c}{1 + cF_i'(0)}[F_i(w) - F(0)]}{F_i(w)},$$

where $F_i(w)$ is any majorant series for the function $f(V_i + v, t)$, and

$$c = \|[J(\tilde{f}, V_i)]^{-1}\|.$$

A close result in the one-dimensional case for a differential equation with polynomial right-hand side was obtained by Pliss (2) by another method.

Example 2. Let V be an n-dimensional vector; $f(V, \lambda)$ a vector-analytic function of V and λ; A a square matrix of order n, of rank $n-1$; $a\,(a_1, a_2, \dots, a_n)$ a vector whose projections are the algebraic complements of the elements of the n-th row of the matrix A (for definiteness it is assumed that $a_n \neq 0$); $b\,(b_1, b_2, \dots, b_n)$ a vector formed by the algebraic complements of the n-th column. Suppose, further, that the equation for the scalar quantity t,

$$b\,f(at, 0) = 0$$

has only simple roots $t_1, t_2, \dots$.

Then between the set $\{t_k\}$ and the set $\{V_k(\lambda)\}$ of all solutions, continuous in a neighborhood of the point 0, of the equation $AV = \lambda f(V, \lambda)$, there exists a one-to-one correspondence. The $V_i(\lambda)$ are representable in the form of power series whose radii of convergence can be estimated; the equality $V_i(0) = a\,t_i$ holds.

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REFERENCES

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2. V. Pliss, DAN, 127, No. 5 (1959).

Note: Figure translations are in progress. See original paper for figures.