N. A. CHERNIKOV

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Abstract

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N. A. CHERNIKOV

THE RELATIVISTIC MAXWELL-BOLTZMANN DISTRIBUTION AND THE INTEGRAL FORM OF THE CONSERVATION LAWS

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In the present work the general form is found of the Einstein metric tensor that admits a distribution of an ideal gas according to the relativistic Maxwell-Boltzmann law. It is proved that such a distribution is possible in a stationary, and only in a stationary, gravitational field if the gas contains particles with nonzero rest mass. If, however, the gas consists entirely of particles with zero rest mass, then a weaker condition is imposed on the gravitational field. In this case the Maxwell-Boltzmann distribution is possible if and only if the Einstein metric tensor differs from a stationary tensor by an arbitrary scalar factor (or does not differ from it at all). A close connection is noted between the Maxwell-Boltzmann distribution and the integral form of the conservation laws. New integral conservation laws are indicated for the case in which the invariant trace of the mass tensor is equal to zero.

An N-component gas is distributed according to the relativistic Maxwell-Boltzmann law if, for each $i = 1, 2, \dots, N$, the distribution function of the i-th component of the gas is equal to

$$A_i(x, P) = a_i(x) \exp\{-(\xi(x), P)\}, \tag{1}$$

where $\xi(x)$ is a vector field in space-time. For this distribution the collision integral ${}^{(1)}I_{ij}$ is equal to zero, and the kinetic equation takes the form

$$\sum_{r=0}^{6} f^r \frac{\partial}{\partial x^r} A_i(x, P) = 0. \tag{2}$$

Substituting expression (2) into equation (1), we obtain that for all values of p^1, p^2, p^3 the equality

$$2p^\alpha \frac{\partial}{\partial x^\alpha} a_i = a_i (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) p^\alpha p^\beta \tag{3}$$

must hold.

Consequently, a_i does not depend on x; if at least one rest mass m_i (m_i is the rest mass of a particle of species i) is not equal to zero, then

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0; \tag{4}$$

whereas if all m_i are equal to zero, then

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 2\varphi(x) g_{\alpha\beta}, \tag{4a}$$

where $\varphi(x)$ is a scalar function. The right-hand side of the last equality arises because in this case $g_{\alpha\beta} p^\alpha p^\beta = 0$.

In addition to condition (4) or (4a), one more condition is imposed on the vector field $\xi(x)$, owing to the requirement of convergence of the integral representing the flux vector [(2)] of particles of species i. In view of this requirement, the vector field $\xi(x)$ must be directed into the interior of the upper half of the light cone, i.e.

$$(\xi(x), \xi(x)) > 0, \qquad \xi_0(x) > 0. \tag{5}$$

Under this condition the integrals representing the flux vector and the mass tensor converge. They can be calculated in the following way. We intro-

we introduce into consideration the Bessel function $K_n(u)$:

$$K_n(u) = \frac{u^n}{1 \cdot 3 \cdot 5 \ldots (2n-1)} \int_0^\infty \exp\{-u \operatorname{sh} s\} \operatorname{sh}^{2n} s \, ds. \tag{6}$$

We have

$$\Phi_i = \int_{\Pi_i} \exp\{-(\xi(x), P)\} \, dP = \frac{4\pi m_i^2 c^2}{u} K_1(u). \tag{7}$$

where

$$u = m_{ic} \sqrt{(\xi(x), \xi(x))}. \tag{8}$$

Consequently, the flux vector of the i-th component is equal to

$$n_i^\alpha = -a_i \frac{\partial \Phi_i}{\partial \xi_\alpha} = \frac{4\pi a_i m_i^4 c^4}{u^2} K_2(u) \xi^\alpha, \tag{9}$$

and the mass tensor of the i-th component is equal to

$$T_i^{\alpha\beta} = a_i \frac{\partial^2 \Phi_i}{\partial \xi_\alpha \partial \xi_\beta} = \frac{4\pi a_i m_i^4 c^4}{u^2} \left\{ \frac{m_i^2 c^2}{u} K_3(u) \xi^\alpha \xi^\beta - K_2(u) g^{\alpha\beta} \right\}. \tag{10}$$

For $m_i = 0$ the formulas are greatly simplified:

$$\Phi_i = \frac{4\pi}{(\xi, \xi)}, \qquad n_i^\alpha = \frac{8\pi a_i}{(\xi, \xi)^2} \xi^\alpha; \tag{9a}$$

$$T_i^{\alpha\beta} = \frac{8\pi a_i}{(\xi, \xi)^2} \left\{ \frac{4\xi^\alpha \xi^\beta}{(\xi, \xi)} - g^{\alpha\beta} \right\}. \tag{10a}$$

The method of integration used here is borrowed from the book ([3]), where the flux vector and the mass tensor are calculated in Galilean coordinates.

It is not difficult to see that the direction of the vector $\xi(x)$ determines the local velocity $\Omega(x)$ of the gas, while its modulus determines the local temperature $\Theta(x)$ of the gas:

$$\sqrt{(\xi(x), \xi(x))} = \frac{c}{k\Theta(x)}, \tag{11}$$

where k is Boltzmann’ s constant.

Let us note that in the case of the Maxwell–Boltzmann distribution, not only the divergence of the mass tensor of the entire gas is equal to zero ([2]), but also the divergence of the mass tensor of each component of the gas.

Equation (4) is called the Killing equation. Equation (4a) is very closely connected with it. Indeed, if $\rho(x)$ obeys the equation

$$\xi^\alpha \frac{\partial \rho}{\partial x^\alpha} = 2\rho\varphi, \tag{12}$$

and the tensor $g_{\alpha\beta}$ obeys equation (4a), then the tensor $h_{\alpha\beta} = \rho^{-1} g_{\alpha\beta}$ obeys equation (4). If the coordinates x^0, x^1, x^2, x^3 are chosen so that the vector field $\xi(x)$ has contravariant components $\{1, 0, 0, 0\}$, then in these coordinates the tensor $g_{\alpha\beta}$ satisfying equation (4) does not depend on x^0. Thus, if not all m_i are equal to zero, then the Maxwell–Boltzmann distribution is possible if and only if in the indicated coordinates the metric tensor $g_{\alpha\beta}$ does not depend on x^0. If, however, all m_i are equal to zero, then the Maxwell–Boltzmann distribution is possible if and only if

$$g_{\alpha\beta} = \rho(x^0, x^1, x^2, x^3) h_{\alpha\beta}(x^1, x^2, x^3).$$

In these coordinates the Maxwell-Boltzmann distribution takes the simplest form:

$$A_i(x,P) = a_i e^{-p_0} = a_i \exp\left\{-\sqrt{m_i^2 c^2 g_{00} + [g_{0k}g_{0l} - g_{00}g_{kl}]\, p^k p^l}\right\}. \tag{13}$$

Let us now consider the question of the connection between the Maxwell-Boltzmann distribution and the integral form of the conservation laws. The integral form

V. A. Fock [4] investigated the integral form of conservation laws. Let the mass tensor $T^{\alpha\beta}(x)$ of some closed system be given. Its divergence is zero. Let $\xi(x)$ be a vector field. V. A. Fock proved that the integral conservation law for the quantity $\eta^\alpha = \xi_\beta T^{\alpha\beta}$ holds if the vector field $\xi(x)$ satisfies condition (4). To this one may add that if the trace $T = T^\alpha_\alpha$ of the mass tensor is equal to zero, then the integral conservation law also holds in the case when the vector field $\xi(x)$ satisfies the weaker condition (4a).

Hence one can draw the following conclusion. If the metric tensor $g_{\alpha\beta}(x)$ admits a gas distribution according to the relativistic Maxwell-Boltzmann law and the rest masses of all particles of the gas are zero, then the integral conservation law is satisfied for the quantity $\xi_\beta T^{\alpha\beta}$, where $T^{\alpha\beta}$ is a mass tensor with zero trace T, for example the mass tensor of the electromagnetic field. If, however, the rest masses of some particles of the gas are not zero and the metric tensor admits the distribution of this gas according to the relativistic Maxwell-Boltzmann law, then the integral conservation law is satisfied for the quantity $\xi_\beta T^{\alpha\beta}$, where $T^{\alpha\beta}$ is an arbitrary mass tensor.

Let us consider equation (4a) in more detail. Denote $\xi_{\alpha\beta} = \nabla_\beta \xi_\alpha$. If the vector field $\xi(x)$ obeys equation (4a), then we have the equality

$$\xi_{\alpha\beta} + \xi_{\beta\alpha} = 2\varphi(x) g_{\alpha\beta}. \tag{14}$$

Let us perform an infinitesimal transformation of the space of events, $x^{\alpha'} = x^\alpha + \xi^\alpha(x) d\lambda$. In view of (14), the interval between infinitely close points is transformed as follows:

$$ds'^2 = (1 + 2\varphi(x) d\lambda)\, ds^2. \tag{15}$$

Thus this transformation is conformal. Conversely, (14) follows from (15).

It turns out that, for a given $g_{\alpha\beta}(x)$, equation (14) has a solution not for every function $\varphi(x)$. Indeed, let $\xi_{\alpha\beta\gamma} = \nabla_\gamma \xi_{\alpha\beta}$, $\varphi_\gamma = \nabla_\gamma \varphi$. Then from (14) it follows that

$$\xi_{\alpha\beta\gamma} = \xi_\nu R^\nu_{\gamma,\alpha\beta} + \varphi_\beta g_{\alpha\gamma} - \varphi_\alpha g_{\beta\gamma} + \varphi_\gamma g_{\alpha\beta}, \tag{16}$$

where $R^\nu_{\gamma,\alpha\beta}$ is the curvature tensor. Thus we obtain a system of equations for 20 functions $\xi_\alpha, \xi_{\alpha\beta}$, connected by the 10 conditions (14). The integrability conditions of these equations are as follows:

$$\xi_\nu R^\nu_{\alpha,\gamma\mu} + \xi_{\alpha\nu} R^\nu_{\beta,\gamma\mu} + \xi_{\nu\mu} R^\nu_{\gamma,\alpha\beta} - \xi_{\nu\gamma} R^\nu_{\mu,\alpha\beta} + \xi_\nu(\nabla_\mu R^\nu_{\gamma,\alpha\beta} - \nabla_\gamma R^\nu_{\mu,\alpha\beta}) =$$
$$= \varphi_{\alpha\mu} g_{\beta\gamma} + \varphi_{\beta\gamma} g_{\alpha\mu} - \varphi_{\alpha\gamma} g_{\beta\mu} - \varphi_{\beta\mu} g_{\alpha\gamma}, \tag{17}$$

where $\varphi_{\mu\nu} = \nabla_\nu \nabla_\mu \varphi = \varphi_{\nu\mu}$. The system of equations under consideration is completely integrable if

$$\delta^\mu_\beta R^\nu_{\alpha,\gamma\sigma} - \delta^\nu_\beta R^\mu_{\alpha,\gamma\sigma} + \delta^\mu_\sigma R^\nu_{\gamma,\alpha\beta} - \delta^\nu_\sigma R^\mu_{\gamma,\alpha\beta} +$$
$$+ \delta^\nu_\alpha R^\mu_{\beta,\gamma\sigma} - \delta^\mu_\alpha R^\nu_{\beta,\gamma\sigma} + \delta^\nu_\gamma R^\mu_{\sigma,\alpha\beta} - \delta^\mu_\gamma R^\nu_{\sigma,\alpha\beta} = 0; \tag{18a}$$

$$\nabla_\mu R^\nu_{\gamma,\alpha\beta} - \nabla_\gamma R^\nu_{\mu,\alpha\beta} = 0; \tag{18b}$$

$$\varphi_{\alpha\sigma} g_{\beta\gamma} + \varphi_{\beta\gamma} g_{\alpha\sigma} - \varphi_{\alpha\gamma} g_{\beta\sigma} - \varphi_{\beta\sigma} g_{\alpha\gamma} = 2\varphi R_{\alpha\beta,\gamma\sigma}. \tag{18c}$$

From (18a) it follows that in this case the space of events is a space of constant curvature K:

$$R_{\mu\nu,\alpha\beta} = K\{g_{\nu\alpha} g_{\mu\beta} - g_{\mu\alpha} g_{\nu\beta}\}. \tag{19}$$

In this case condition (18b) is satisfied identically, while condition (18c) is written in the form

$$\varphi_{\alpha\sigma} g_{\beta\gamma} + \varphi_{\beta\gamma} g_{\alpha\sigma} - \varphi_{\alpha\gamma} g_{\beta\sigma} - \varphi_{\beta\sigma} g_{\alpha\gamma} = 0, \tag{20}$$

where $\psi_{\alpha\beta} = \varphi_{\alpha\beta} - K\varphi g_{\alpha\beta}$. Multiplying (20) by $g^{\sigma\mu} g^{\gamma\nu}$ and summing over σ and over γ, we obtain an equality from which it is easy to derive that $\psi_{\alpha\beta} = 0$, and consequently,

$$\nabla_\alpha \nabla_\beta \varphi = K\varphi g_{\alpha\beta}. \tag{21}$$

Let us consider the case of special relativity, when $K = 0$. In this case one may choose coordinates x^0, x^1, x^2, x^3 in which $g_{\alpha\beta} = \text{const}$. From (21) it follows that in these coordinates

$$\varphi(x) = a + (b, x), \tag{22}$$

where a is a constant scalar and b is a constant vector. The vector field $\xi(x)$ depends on 15 parameters and has the form

$$\xi^\alpha = \varphi(x)x^\alpha - \frac{1}{2}(x,x)b^\alpha + g^{\alpha\beta}\eta_{\beta\gamma}x^\gamma + \eta^\alpha, \tag{23}$$

where $\eta_{\beta\gamma}+\eta_{\gamma\beta} = 0$, and η^α and $\eta_{\beta\gamma}$ are constants of integration. The vector field (23) is represented as a linear combination of vector fields with coefficients η, a, and b. The vector fields with coefficients η give the usual integral conservation laws for momentum and angular momentum ([4]). The vector fields with a and b give the conservation law for angular momentum with respect to a conformal transformation of space-time in the case when the trace of the mass tensor is equal to zero.

An analogous study of equations (4) was carried out in ([4], [5]).

The results obtained make it possible to pose the problem of determining the gravitational field produced by a gravitating gas distributed according to the relativistic Maxwell-Boltzmann law. The problem reduces to solving the gravitational equations

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = -\frac{8\pi\gamma}{c}\sum_{i=1}^{N} T_i^{\alpha\beta}, \tag{24}$$

where $T_i^{\alpha\beta}$ is the tensor (10) or (10a), and if at least one mass m_i is not equal to zero, then the vector field $\xi(x)$ satisfies conditions (4) and (5), while if all m_i are equal to zero, then the field $\xi(x)$ satisfies conditions (4a) and (5). In this case the kinetic equation is satisfied. Let us emphasize that in this problem the mass tensor $T^{\alpha\beta}(x)$ is not given, but is determined together with the metric tensor. This problem is a particular case of the problem posed at the end of paper ([2]).

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United Institute
for Nuclear Research

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Note: Figure translations are in progress. See original paper for figures.