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Abstract

Full Text

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On a Class of Nonlinear Degenerate Parabolic Equations

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In a number of questions in mechanics, in problems of heat conduction, filtration, and the boundary layer, nonlinear equations are encountered whose order or type depends on the value of the unknown function. An example of such equations is the equation

$$u_{xx} - a(u)u_t = 0 \quad \text{or} \quad u_{xx} - (A(u))_t = 0, \tag{1}$$

where $\infty > a > 0$ for $\infty > u > 0$, $a(0) = 0$, $\infty > a' > 0$ for $\infty > u > 0$. We shall prove the existence and uniqueness of a generalized solution of the first boundary-value problem and of the Cauchy problem, study the question of the existence of a classical solution of these problems, and carry out a qualitative investigation of the solutions.

1. Consider the first boundary-value problem for equation (1) in the domain $G\{0 \leq x \leq X,\ t \geq 0\}$

$$u\big|_{t=0} = u_0(x) \geq 0, \qquad u\big|_{x=0} = \varphi_1(t) \geq 0, \qquad u\big|_{x=X} = \varphi_2(t) \geq 0. \tag{2}$$

Let all the functions $u_0(x)$, $\varphi_1(t)$, $\varphi_2(t)$ have one bounded derivative, and let $u_0(0) = \varphi_1(0)$, $u_0(X) = \varphi_2(0)$.

A bounded continuous function $u(x,t)$ is called a generalized solution of the first boundary-value problem (1)–(2) if it satisfies conditions (2), has a generalized derivative with respect to x, and for every smooth finite function $f(x,t)$ whose support lies strictly inside G, the identity

$$\iint \left(\frac{\partial f}{\partial t} A(u) - \frac{\partial f}{\partial x}\frac{\partial u}{\partial x} \right) dx\, dt = 0 \tag{3}$$

holds.

Theorem 1. *The generalized solution of problem (1)–(2) exists and is unique.*

Consider the sequence of equations

$$u^\varepsilon_{xx} - (a(u^\varepsilon) + \varepsilon)u_t = 0, \quad \varepsilon > 0 \quad \text{or} \quad u^\varepsilon_{xx} - \left(A(u^\varepsilon) + \varepsilon u^\varepsilon\right)_t = 0, \tag{4}$$

with initial and boundary data (2). As shown in ([1]), these equations have classical solutions, which form, by virtue of the maximum principle, a uniformly bounded sequence $\{u_\varepsilon\}$. The derivatives with respect to x, u^ε_x, are also uniformly bounded. This can be proved in the same way as in ([3]).

We now show that the family of functions $\{u^\varepsilon\}$ is equicontinuous. From this will follow compactness in the sense of uniform convergence and the existence of a generalized derivative with respect to x for the limiting function. Integrating equation (4) over a narrow strip $G_{\Delta t}\{0 \le x \le X,\ t_0 \le t \le t_0 + \Delta t\}$, it is easy to obtain, by virtue of the uniform boundedness of u^ε and u^ε_x (S_τ is the line $t = \tau$),

$$\int_{S_t} A(u^\varepsilon)\,dx - \int_{S_{t+\Delta t}} A(u^\varepsilon)\,dx = O(\varepsilon) + O(\Delta t).$$

Since $A(u)$ is a monotone continuous function, it suffices to prove the uniform continuity of the functions $A(u^\varepsilon)$.

Let (x_0, t_0) be an arbitrary point, and let δ be so small that for all ε, t, and $x^*, x^{**} \in [x_0 - \delta/2,\ x_0 + \delta/2]$ one has

$$|A(u^\varepsilon(x^*, t)) - A(u^\varepsilon(x^{**}, t))| < \eta/2.$$

Next choose Δt and ε so that the integral from $x_0 - \delta/2$ to $x_0 + \delta/2$ of the function

$$|A(u^\varepsilon(x, t + \Delta t)) - A(u^\varepsilon(x, t))|$$

is less than $\eta\delta/2$. Since the integrand is continuous, there exists a point $\bar{x}(\varepsilon)$ such that

$$|A(u^\varepsilon(\bar{x}, t + \Delta t)) - A(u^\varepsilon(\bar{x}, t))| < \eta\delta/2\delta = \eta/2,$$

and then

$$|A(u^\varepsilon(x_0 + \Delta x, t + \Delta t)) - A(u^\varepsilon(x_0, t))| < 2\eta.$$

From all that has been said it follows that in the integral identity (3), written for u^ε, one may pass to the limit along some subsequence u^{ε_k}, and this limit is a continuous function (x, t). Let us show that the equation is satisfied almost everywhere. Multiply equation (4) by u^ε_t and integrate over the domain G. Then, after elementary transformations, we obtain

$$\iint \frac{[A(u^\varepsilon)]^2_t}{a(u^\varepsilon)}\,dx\,dt + \frac{1}{2}\int_{S_t} (u^\varepsilon_x)^2\,dx + \varepsilon \iint (u^\varepsilon_t)^2\,dx\,dt = \int_0^T u_{tu} x\,dx\Big|_{x=0}^{x=X} + \frac{1}{2}\int_{S_0} (u^\varepsilon_x)^2\,dx. \tag{5}$$

Now square equation (4) and integrate over the domain:

$$\iint (u^\varepsilon_{xx})^2\,dx\,dt = \iint [A(u^\varepsilon)]^2_t\,dx\,dt + 2\iint \varepsilon a(u^\varepsilon) u^\varepsilon_t\,dx\,dt + \iint \varepsilon^2 (u^\varepsilon_t)^2\,dx\,dt.$$

From (5) and the uniform boundedness of $a(u^\varepsilon)$ there follows the uniform boundedness of the right-hand side of the written identity, i.e. $\|(A(u^\varepsilon)+\varepsilon u^\varepsilon)_t\|_{L_2}$ is uniformly bounded in ε, and therefore the limiting function has the generalized derivative $(A(u))_t$, and in equation (4) passage to the weak limit is possible.

Uniqueness of the generalized solution is proved in the same way as in ([2]). At points where $u \neq 0$, just as in ([1]), with the aid of Bernstein estimates it is proved that the solution is classical, i.e. has the continuous derivatives entering into the equation, if $a(u)$ is a sufficiently smooth function. If the solution vanishes at some point, then it may have a discontinuous derivative with respect to t at that point. Indeed, consider the equation

$$u_{xx} - u^\alpha u_t = 0, \qquad \alpha > 0. \tag{6}$$

We construct the solution in the form

$$u = (1-t)^{1/\alpha}\varphi(x), \qquad \varphi''(x) + \frac{1}{\alpha}\varphi^{\alpha+1} = 0.$$

If $\varphi(x)$ is a solution of this equation with initial data $\varphi(0)=1$, $\varphi'(0)=0$, then $\varphi(x)$ is a convex function equal to 0 at certain points $|x| = h$. We shall consider the solution $u(x,t)$ in the domain $G_h\{-h \le x \le h,\ 0 \le t \le 2\}$. It is easy to verify that the function

$$u(x,t) = \begin{cases} (1-t)^{1/\alpha}\varphi(x), & t \le 1, \\ 0, & t \ge 1, \end{cases}$$

is a generalized solution of equation (6) with data $u(x,0) = \varphi(x)$, $u(h,t) = u(-h,t) = 0$. For $\alpha < 1$ the solution has a continuous derivative with respect to t; for $\alpha = 1$, $u_t(x,t)$ has a discontinuity of the first kind; for $\alpha > 1$, $u_t(x,t) = \infty$.

2. Qualitative study of the generalized solution

Theorem 2. *If a generalized solution of equation (1) vanishes at some point (η, δ), then it vanishes at once on the entire straight line $t = \delta$.*

Proof. Suppose this is not so, i.e., $u(\eta,\delta) = 0$, but there is a point on the line $t = \delta$ where $u \neq 0$. For simplicity let this point have coordinates $(0,\delta)$. In this case there exists a μ such that all the functions $u^\varepsilon(0,\delta) > \mu$, starting with some ε. By the property of equicontinuity of the functions u^ε, this is true for the functions u^ε on some segment of the t-axis. For simplicity let the length of this segment be δ. Consider the function $v = \beta(at+x)^2$ in the domain G_a bounded by the straight lines $x + at = 0$, $x = 0$, $t = \delta$. If β is sufficiently small, then $a(v)$ is also small and $L_\varepsilon v > 0$. If β is still so small that $v - u_\varepsilon < 0$ for $x = 0$, then $v - u_\varepsilon < 0$ everywhere in G_a, since $L_\varepsilon(v - u_\varepsilon) > 0$. But then $u(\eta,\delta) \neq 0$.

Theorem 3. *The solution of the first boundary-value problem (1)–(2) with initial and boundary data $\varphi_1(t) = \varphi_2(t) = 0$, $u_0(x) \ge 0$, $u_0(x) \not\equiv 0$ vanishes at* *interior points of G if and only if the integral from 0 to 1 of the function $a(u)/u$ converges.*

Proof. We show that if the indicated integral converges, then for any τ the functions $|u^\varepsilon| < (X+1)^2\tau$, starting with some $t = t_0$ (t_0 does not depend on τ) for all $\varepsilon > \varepsilon_0(\tau)$. Introduce the function $\psi(t)$ by the equality

$$\gamma \int_\varepsilon^{\psi(t)} \frac{a(\beta u)}{u}\, du = t_0 - t, \quad t \le t_0,$$

and consider the function

$$v = (k - x^2)\psi(t),$$

where $k = (X+1)^2$, and β is such that $k - x^2/\beta < 1$. We have

$$L_\varepsilon v = \psi(t)\left[-2 + (a((k-x^2)\psi) + \varepsilon)\frac{(k-x)^2}{\gamma a(\beta\psi)}\right].$$

The function $\psi(t) \ge \tau$; therefore ε can be chosen so that $a((k-x^2)\psi) > \varepsilon$, and γ so that $L_\varepsilon v < 0$. If, moreover, t_0 is chosen independently of τ so that $\psi|_{t=0} > u_0(x)/(k-x^2)$ (and this is possible, since the integral converges), then, by the maximum principle, $u^\varepsilon < v$ everywhere inside G, and therefore $u = 0$ for $t \ge t_0$.

Now we show that if the integral indicated in the hypotheses of the theorem diverges, then the solution $u(x,t)$ nowhere inside G vanishes.

Introduce the function $\psi(t)$ as follows:

$$\gamma \int_{\psi(t)}^{1} \frac{a(\beta u)}{u}\, du = t, \quad 0 \le t < \infty.$$

$1 \ge \psi > 0$, and consider the function $v = \varphi(x)\psi(t)$, where $\varphi(x)$ has the following form. Let $[x_1, x_2]$ be an interval on which $u_0(x) \ge \chi > 0$, $0 < \varphi(x) < \chi$, $\varphi(x_1) = \varphi(x_2) = 0$ for $x \in [x_1, x_2]$; $\varphi''(x) \ge 0$ and is bounded when $\varphi \le \chi/2$, $\varphi''(x) \le 0$ when $\varphi \ge \chi/2$,

$$L_\varepsilon v = \psi(t)\left[\varphi''(x) + (a(\varphi\psi) + \varepsilon)\frac{\varphi}{\gamma a(\beta\psi)}\right]$$

there, where $\varphi'' > 0$, $L_\varepsilon v > 0$. Choose β so that $\chi/2\beta > 1$; choose γ so that $L_\varepsilon v > 0$ when $\varphi \ge \chi/2$ ($\varphi'' \le 0$). From the maximum principle it follows that $u^\varepsilon > v$ everywhere inside the domain G.

3. Existence and uniqueness of the classical solution of the Cauchy problem

By the same method by which the existence of a solution of the first boundary-value problem was proved, one can prove the existence of a generalized solution of the Cauchy problem for $t \geq 0$ with the condition

$$u|_{t=0} = u_0(x) \geq 0, \quad |u_0'(x)| \leq M.$$

We shall show that the generalized solution of the Cauchy problem does not vanish if $u_0(x)$ is finite and $u_0(x) \not\equiv 0$. Hence it follows easily that the constructed solution of the Cauchy problem for any $u_0(x) \not\equiv 0$ is different from zero for $t > 0$ and therefore, as was indicated above, is always classical. This qualitatively distinguishes equation (1) from the equation of nonstationary filtration, where, as is known, the zero of a solution propagates with finite speed. We shall construct the solution of the Cauchy problem as the limit of solutions u^ε of the first boundary-value problem in the domain G with the conditions

$$u^\varepsilon(x,0)\big|_{|x|\leq 1/\varepsilon - 1} = u_0(x), \quad u^\varepsilon(x,0)$$

is continuous, nonnegative, and

$$|u^\varepsilon(x,0)| \leq \max|u_0(x)|, \quad u^\varepsilon(1/\varepsilon, t) = u^\varepsilon(-1/\varepsilon, t) = 0, \quad |u^\varepsilon| \leq M.$$

As was shown in Theorem 1, the functions u^ε are equicontinuous in x and t and have x-derivatives uniformly bounded with respect to ε. It is not difficult to see that the limiting function is a generalized solution of the Cauchy problem for equation (1). The uniqueness of this solution is proved analogously to how this was done in [2].

Let us show that $\|u_t^\varepsilon\|_{E_2}$ are uniformly bounded. Indeed, multiply (4) by u_t^ε and integrate over the domain $G_{1/\varepsilon}$:

$$\iint_{G_{1/\varepsilon}} (a(u^\varepsilon) + \varepsilon)(u_t^\varepsilon)^2 \, dx \, dt + \frac{1}{2}\int_{S_T} (u_x^\varepsilon)^2 \, dt = \frac{1}{2}\int_{S_0} (u_x^\varepsilon)^2 \, dt.$$

Suppose that the generalized solution of the Cauchy problem vanishes at $t = T$. It is easy to show that the following formula is valid:

$$\int_0^T u_x(N,t)\, dt - \int_0^T u_x(-N,t)\, dt = \int_{-N}^N A(u(x,0))\, dx - \int_{-N}^N A(u(x,T))\, dx.$$

But $A(u(x,T)) = 0$, and there exist such values of N (since the function u_x is square-summable) that the terms on the left-hand side are arbitrarily small. The contradiction obtained shows that the solution $u(x, t)$ cannot vanish at $t > 0$ if $u_0(x) \not\equiv 0$.

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Note: Figure translations are in progress. See original paper for figures.