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1962

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Abstract

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ON THE THEORY OF BEST APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY ENTIRE FUNCTIONS OF FINITE DEGREE

(Presented by Academician A. N. Kolmogorov on 28 VIII 1961)

The notion of best uniform approximation of functions of one and several variables by entire functions of finite degree was first given by S. N. Bernstein, who also completely solved a number of problems connected with this question. At the present time the theory of best uniform and mean approximation of functions by entire functions of finite degree has been enriched by many interesting results, chiefly in the works of Soviet mathematicians. Thus, for example, S. M. Nikol' skii ([1,2]) investigated the order of the best approximation of functions from the class $H_p^{(r_1,\dots,r_n)}[M]$ and applied the results obtained to the theory of differentiable functions.

Below we present theorems on the best approximation of functions of several variables by entire functions of finite spherical degrees, which are generalizations of the corresponding known results.

Let E_n be the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$; let $n_1, \dots, n_k$ be natural numbers whose sum is equal to n, and let $E^{(i)}$ be the n_i-dimensional spaces of points $x^{(i)} = (x_1^{(i)}, \dots, x_{n_i}^{(i)})$ $(i = 1, \dots, k)$. Therefore one may consider that $E_n = E^{(1)} \times \dots \times E^{(k)}$, $x = (x^{(1)}, \dots, x^{(k)})$. Further, if $x, y \in E_n$, then $x \pm y = (x_1 \pm y_1, \dots, x_n \pm y_n)$; if α is a real number, then $\alpha x = (\alpha x_1, \dots, \alpha x_n)$ and

$$|x| = \left(\sum_1^n x_s^2 \right)^{1/2}.$$

In what follows we shall assume that $\varphi(x)$ is continuous in E_n, $\varphi(x) > 0$, and that for it

$$\int_0^\infty \frac{\ln \alpha(t)\, dt}{1 + t^2} < \infty, \quad \text{where} \quad \alpha(t) = \sup_{\substack{x,\xi \in E_n \\ |\xi| \le t}} \frac{\varphi(x + \xi)}{\varphi(x)}$$

(this condition was introduced in ([3,4])).

Let $L_{p,\varphi}^{(n)}$ be the space of functions $f(x)$, defined and measurable in E_n, for which

$$\left|\frac{f(x)}{\varphi(x)}\right|^p$$

is Lebesgue integrable over E_n, and the norm is defined by the equality

$$\|f\|_{p,\varphi}^{(n)} = \left(\int_{E_n}\left|\frac{f(x)}{\varphi(x)}\right|^p dx\right)^{1/p} \quad (1 \le p < \infty).$$

As usual, it is meant that $f(x) \in L_{\infty,\varphi}^{(n)}$ if $\dfrac{f(x)}{\varphi(x)}$ is essentially bounded, and in this case the norm is defined as follows:

$$\|f\|_{\infty,\varphi}^{(n)} = \sup_{x\in E_n} \operatorname{vrai}\left|\frac{f(x)}{\varphi(x)}\right|.$$

It is obvious that $L_{p,\varphi}^{(n)}$ $(1 \le p \le \infty)$ is a Banach space.

Let s be a natural number, $h \in E^{(i)}$, and

$$\Delta_{hx^{(i)}}^s f = \sum_{j=0}^{s}(-1)^{s-j}\binom{s-j}{j} f\big(x^{(1)},\dots,x^{(i-1)},x^{(i)}+jh,x^{(i+1)},\dots,x^{(k)}\big).$$

$\omega_{sx^{(i)}}(\delta,f)_{p,\varphi}^{(n)}$ $(0 \le \delta < \infty)$ will be called the modulus of continuity of order s of the function $f(x) \in L_{p,\varphi}^{(n)}$ with respect to $x^{(i)}$, if

$$\omega_{sx^{(i)}}(\delta,f)_{p,\varphi}^{(n)} = \sup_{|h|\le\delta}\frac{1}{\alpha_i(s\delta)}\left\|\Delta_{hx^{(i)}}^s f\right\|_{p,\varphi}^{(n)},$$

where

$$\alpha_i(t) = \sup_{\substack{x\in E_n,\ \xi\in E^i\\ |\xi|\le t}}\frac{\varphi(x^{(1)},\dots,x^{(i-1)},x^{(i)}+\xi,x^{(i+1)},\dots,x^{(k)})}{\varphi(x)}$$

(analogous notions for other spaces were introduced earlier).

Suppose that for $s = 1,\dots,n_i$ $f(x)$ has, in the generalized sense of S. L. Sobolev, the derivative $\partial f(x)/\partial x_s^{(i)}$. Further, let l_i be any direction in $E^{(i)}$, and let $\xi_1^{(i)},\dots,\xi_{n_i}^{(i)}$ be its direction cosines in $E^{(i)}$. Then the expression

$$\frac{f(x)}{\partial l_i} = \sum_{s=1}^{n_i} \xi_s^{(i)} \frac{\partial f(x)}{\partial x_s^{(i)}}$$

will be called the derivative of the function $f(x)$ in the direction l_i, and

$$\frac{\partial^r f(x)}{\partial l_i^r} = \frac{\partial}{\partial l_i}\left[\frac{\partial^{r-1} f(x)}{\partial l_i^{r-1}}\right] \qquad (r = 2, 3, \dots)$$

will be called the derivative of order r of the function $f(x)$ in the direction l_i.

We note that if $f(x) \in L_{p,\varphi}^{(n)}$ and, for every direction l_i in $E^{(i)}$, $\partial^r f(x)/\partial l_i^r \in L_{p,\varphi}^{(n)}$, then

$$\omega_{r,x^{(i)}}(\delta, f)_{p,\varphi}^{(n)} \le \delta^r \sup_{l_i} \left\|\frac{\partial^r f}{\partial l^r}\right\|_{p,\varphi}^{(n)},$$

$$\omega_{s+r,x^{(i)}}(\delta, f)_{p,\varphi}^{(n)} \le \delta^r \alpha_i(s\delta) \sup_{l_i} \omega_{sx^{(i)}}\left(\delta, \frac{\partial^r f}{\partial l_i^r}\right)_{p,\varphi}^{(n)}.$$

Let $f \in L_{p,\varphi}^{(n)}$. We shall consider, along with it, all possible $g_{\nu x^{(i)}}$ and $g_{\nu_1 x^{(1)},\dots,\nu_{kx}^{(k)}}$, also belonging to $L_{p,\varphi}^{(n)}$. The function $g_{\nu x^{(i)}}$ has the property that it is an entire function of the aggregate of variables $(x_1^{(i)}, \dots, x_{n_i}^{(i)})$ and has spherical degree $\le \nu$, i.e., for almost all fixed $x^{(1)}, \dots, x^{(i-1)}, x^{(i+1)}, \dots, x^{(k)}$, for every $\varepsilon > 0$ there exists $A > 0$ such that for all complex $z^{(i)} = (z_1^{(i)}, \dots, z_{n_i}^{(i)})$ the inequality

$$\left|g_{\nu x^{(i)}}(x^{(1)}, \dots, x^{(i-1)}, z^{(i)}, x^{(i+1)}, \dots, x^{(k)})\right| \le A e^{(\nu+\varepsilon)|z^{(i)}|^*},$$

holds, where

$$|z^{(i)}|^* = \sqrt{|z_1^{(i)}|^2 + \dots + |z_{n_i}^{(i)}|^2},$$

and $g_{\nu_1 x^{(1)},\dots,\nu_{kx}^{(k)}}$ has the property that it is an entire function in the aggregate of all arguments and, for $\varepsilon > 0$, there exists $A > 0$ such that for all complex $z^{(1)}, \dots, z^{(k)}$ the inequality

$$\left|g_{\nu_1 x^{(1)},\dots,\nu_{kx}^{(k)}}(z^{(1)}, \dots, z^{(k)})\right| \le A \exp\left[\sum_{i=1}^{k} (\nu_i + \varepsilon)|z^{(i)}|^*\right].$$

We shall call the greatest lower bound

$$A_{\nu x^{(i)}}(f)^{(n)}_{p,\varphi} = \inf_{g_{\nu x^{(i)}}} \|f - g_{\nu x^{(i)}}\|^{(n)}_{p,\varphi},$$

extended over all possible functions $g_{\nu x^{(i)}}$, the best approximation of the function f by means of $g_{\nu x^{(i)}}$, and the best approximation of the function f by means of entire functions $g_{\nu_1 x(1),\dots,\nu_k x(k)}$—the lower bound

$$A_{\nu_1 x(1),\dots,\nu_k x(k)}(f)^{(n)}_{p,\varphi} = \inf_{g_{\nu_1 x(1),\dots,\nu_k x(k)}} \left\|f - g_{\nu_1 x(1),\dots,\nu_k x(k)}\right\|^{(n)}_{p,\varphi},$$

extended over all possible $g_{\nu_1 x(1),\dots,\nu_k x(k)}$.

Theorem 1. Let s_i $(i = 1, \dots, k)$ be natural numbers and let the function $f(x) \in L^{(n)}_{p,\varphi}$ have derivatives $\partial^{r_i} f/\partial l_i^{r_i} \in L^{(n)}_{p,\varphi}$ $(r_i \geqslant 0)$, where l_i is an arbitrary direction in $E^{(i)}$ $(i = 1, \dots, k)$. Then

$$A_{\nu_1 x(1),\dots,\nu_k x(k)}(f)^{(n)}_{p,\varphi} \leqslant d \sum_{i=1}^{k} \frac{1}{\nu_i^{r_i}} \sup_{l_i} \omega_{s_i x(i)} \left(\frac{1}{\nu_i}, \frac{\partial^{r_i} f}{\partial l_i^{r_i}}\right)^{(n)}_{p,\varphi},$$

where $\nu_i \geqslant 1$; d does not depend on $f, \nu_1, \dots, \nu_k$.

The following theorems are converses with respect to Theorem 1.

Theorem 2. Let $s_i, \nu_i \leqslant \delta^{-1}$ be natural numbers, $0 < \alpha_i < s_i$, $\psi_i(\delta) \in N^{\alpha_i}$ (the last condition was introduced in ([5])), $i = 1, \dots, k$. Then, if

$$A_{\nu_1 x(1),\dots,\nu_k x(k)}(f)^{(n)}_{p,\varphi} \leqslant \sum_{i=1}^{k} M_i \psi_i \left(\frac{1}{\nu_i + 1}\right),$$

then

$$\omega_{s_i x(i)}(\delta, f)^{(n)}_{p,\varphi} \leqslant \left[c_2 M_i + c_2 \|f\|^{(n)}_{p,\varphi}\right] \psi_i(\delta).$$

Theorem 3. Let r be a natural number and let, for the function f, the series

$$\sum_{j=1}^{\infty} j^{r-1} A_{j-1,x(i)}(f)^{(n)}_{p,\varphi}$$

converge. Then for any natural s, $\nu \leqslant \delta^{-1}$, the inequality holds

$$\omega_{sx(i)} \left(\delta, \frac{\partial^r f}{\partial l_i^r}\right)^{(n)}_{p,\varphi} \leqslant c_4 \left\{\nu^{-(r+s)} \sum_{j=1}^{\nu} j^{r+s-1} A_{j-1,x(i)}(f)^{(n)}_{p,\varphi} + \right.$$

$$+\nu^{-s}\|f\|_{p,\varphi}^{(n)}+\sum_{j=\nu+1}^{\infty} j^{r-1}A_{j-1,x(i)}(f)_{p,\varphi}^{(n)}\Bigg\}$$

where c_4 also does not depend on the direction l_i.

Remark. Under various assumptions concerning $n, k, p, \varphi, s, \omega_{sx(i)}(\delta, f)_{p,\varphi}^{(n)}$, the corresponding results from the stated theorems were proved in ($^{1-9}$) in special cases.

If $\nu_{k_1}=\nu_{k_2}=\dots=\nu_{k_m}=\infty$ $(m<k)$, while the remaining $\nu_i<\infty$ $(i=1,\dots,k)$, then below by $G_{\nu_1 x(1),\dots,\nu_k x(k)}(x,\dots,x^{(k)})$ we shall understand, for almost all $x^{(k_1)},\dots,x^{(k_m)}$, an entire function in each of the remaining $x^{(i)}$ of spherical degree ν_i.

For the study of properties of differentiable functions in the norms of the space $L_{p,\varphi}^{(n)}$ by methods of the theory of best approximation of functions, in addition to the theorems indicated above, the following theorem is also needed; it is a generalization of Theorem 1 and of the corresponding theorem of S. M. Nikol' skii (^{2}).

Theorem 4. Let s_i $(i=1,\dots,k)$ be natural numbers and let the functions

$$f(x^{(1)},\dots,x^{(k)})\in L_{p_i,\varphi}^{(n)}\quad (i=1,\dots,k)$$

have derivatives $\partial^{r_i}f/\partial l_i^{r_i}\in L_{p_i,\varphi}^{(n)}$ $(i=1,\dots,k)$, where $r_i\geqslant 0$, l_i is an arbitrary direction in $E^{(i)}$ $(i=1,\dots,k)$. Then there exists a system of functions $G_{\nu_1 x(1),\dots,\nu_k x(k)}\in L_{p_i,\varphi_i}^{(n)}$ $(1\leqslant\nu_i\leqslant\infty,\ i=1,\dots,k)$,

for which the inequalities

$$\left\|f-G_{\nu_1 x^{(1)},\,\infty x^{(2)},\dots,\infty x^{(k)}}\right\|_{p_1,\varphi}^{(n)}\leqslant\frac{d}{\nu_1^{r_1}}\sup_{l_1\in E^{(1)}}\omega_{s_1 x^{(1)}}\left(\frac{1}{\nu_1};\,\frac{\partial^{r_1}f}{\partial l_1^{r_1}}\right)_{p_1,\varphi}^{(n)},$$

$$\left\|G_{\nu_1 x^{(1)},\,\infty x^{(2)},\dots,\infty x^{(k)}}-G_{\nu_1 x^{(1)},\,\nu_2 x^{(2)},\,\infty x^{(3)},\dots,\infty x^{(k)}}\right\|_{p_2,\varphi}^{(n)}\leqslant$$

$$\leqslant\frac{d}{\nu_2^{r_2}}\sup_{l_2\in E^{(2)}}\omega_{s_2 x^{(2)}}\left(\frac{1}{\nu_2};\,\frac{\partial^{r_2}f}{\partial l_2^{r_2}}\right)_{p_2,\varphi}^{(n)},$$

. . . .

$$\left\|G_{\nu_1 x^{(1)},\dots,\nu_{k-1}x^{(k-1)},\,\infty x^{(k)}}-G_{\nu_1 x^{(1)},\dots,\nu_k x^{(k)}}\right\|_{p_k,\varphi}^{(n)}\leqslant$$

$$\leqslant \frac{d}{\nu_k^{r_k}} \sup_{l_k \in E^k} \omega_{s_k x^{(k)}} \left(\frac{1}{\nu_k}, \frac{\partial^{r_k} f}{\partial l_k^{r_k}} \right)_{p_k, \varphi}^{(n)},$$

and other similar inequalities, which can be obtained by permuting the places of $x^{(1)}, \dots, x^{(k)}$ (and correspondingly $\nu_1, \dots, \nu_k$). In the last inequalities ν_i denote finite numbers ($\geqslant 1$).

Moreover, the relation

$$\left\| G_{\nu_1 x^{(1)}, \dots, \nu_k x^{(k)}} \right\|_{p_i, \varphi}^{(n)} \leqslant d \|f\|_{p_i, \varphi}^{(n)}$$

holds.

The constant d everywhere does not depend on f and $\nu_i \geqslant 1$ ($i = 1, \dots, n$).

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Received
21 VII 1961

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