A. I. KOSHELEV

1962

SovietRxiv

Abstract

Full Text

A. I. KOSHELEV

ON THE CONVERGENCE OF THE METHOD OF SUCCESSIVE APPROXIMATIONS FOR QUASILINEAR ELLIPTIC EQUATIONS

(Presented by Academician I. N. Vekua, 11 X 1961)

Let us consider, inside a finite n-dimensional domain Ω bounded by a surface Γ, the differential equation

$$\sum_{i=1}^{n} \frac{\partial a_i(x; u, p_j)}{\partial x_i} - a_0(x; u, p_j) = 0, \tag{1}$$

where u is the unknown function, $x \in \Omega$, and $p_j = \partial u / \partial x_j$ $(j = 1, \dots, n)$. We shall assume that the result of substituting any function $u \in W_2^{(1)}(\Omega)$ into a_i $(i = 0, 1, \dots, n)$ gives functions from $L_2(\Omega)$, and, moreover, that the partial derivatives a_i $(i = 0, 1, \dots, n)$ with respect to the variables $u, p_1, \dots, p_n$ are bounded for all values of x, u, and p. Thus, for any $u, p_1, \dots, p_n$ and $x \in \Omega$, the inequalities

$$\left| \frac{\partial a_i}{\partial p_j} \right| < K \qquad (i, j = 0, 1, \dots, n), \tag{2}$$

hold, where K is some positive constant (which may be sufficiently large) and $\partial / \partial p_0 = \partial / \partial u$.

We shall also assume that, for all x, u, and p_j, and for arbitrary real $\xi_0, \xi_1, \dots, \xi_n$, the inequality

$$\sum_{i,k=0}^{n} \frac{\partial a_i}{\partial p_k} \xi_i \xi_k \geq \alpha \sum_{i=0}^{n} \xi_i^2, \tag{3}$$

holds, where α is some positive constant. Inequality (3) ensures the ellipticity of equation (1).

1. We shall first seek a generalized solution of equation (1) satisfying the boundary condition

$$u\big|_{\Gamma} = 0. \tag{4}$$

By a generalized solution of problem (1)—(4) we mean a function

$$u \in \overset{0}{W}{}_2^{(1)}(\Omega)^*,$$

which, for every function v from the same space, satisfies the equality

$$\int_\Omega \left[\sum_{i=1}^n a_i(x; u, p_j)\frac{\partial v}{\partial x_i} + a_0(x; u, p_j)v\right] dx = 0. \tag{5}$$

Numerous works have been devoted to the study of equation (1) with the boundary condition (4), as well as with some other boundary conditions. In these works not only existence and uniqueness theorems have been obtained, but also the differential properties of solutions have been studied. In the present note we shall show that the generalized solution of problem (1), (4), as well as the re- solutions of the second and third boundary-value problems can be obtained by the ordinary process of successive approximations, starting from the solution of the corresponding boundary-value problem for the Poisson equation. In doing so, we do not impose restrictive conditions of closeness of the operator on the left-hand side of equation (1) to a linear operator. It turns out that conditions (2) and (3) guarantee convergence of the indicated process in the energy metric. Hence, in particular, existence and uniqueness theorems will follow.

Inequalities (2) and (3) can be written in the following form

$$\beta\sum_{i=0}^n \xi_i^2 \ge \sum_{i,k=0}^n \frac{\partial a_i}{\partial p_k}\xi_i\xi_k \ge \alpha\sum_{i=0}^n \xi_i^2, \tag{6}$$

where α and β are positive constants.

In the case when the coefficients a_i $(i = 0, 1, \dots, n)$ do not depend on u, inequalities (6) are consequences of the ellipticity inequality (2) for $\xi_0 = 0$ and of conditions (3) (boundedness of the derivatives of the coefficients a_i).

Let us write equality (5) in the form

$$\int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + u\cdot v\right) dx = \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + u\cdot v\right) dx - \frac{1}{\beta}\int_\Omega \left[\sum_{i=1}^n a_i\frac{\partial v}{\partial x_i} + a_0 v\right] dx. \tag{7}$$

---

* By $\overset{0}{W}{}_2^{(1)}$ we denote the space of functions having all generalized partial derivatives of first order in the sense of S. L. Sobolev and satisfying condition (4).

On the left in the last equality stands the scalar product (u, v) of functions in the space $W_2^{(1)}$.

The right-hand side of equality (7) may be regarded as a linear functional defined on $\overset{0}{W}{}_2^{(1)}$. By Riesz' theorem this functional can be represented in the form (u^*, v), where u^* is some element of $\overset{0}{W}{}_2^{(1)}$. In other words, an operator A is defined which assigns to each $u \in \overset{0}{W}{}_2^{(1)}$ the corresponding element $u^* \in \overset{0}{W}{}_2^{(1)}$. Thus, the problem of finding a generalized solution of problem (1), (4) reduces to solving the equation

$$u = Au.$$

It is easy to see that the operator A is a contraction operator. Indeed, let u_1 and u_2 be arbitrary elements of $\overset{0}{W}{}_2^{(1)}(\Omega)$, and let $w = u_1 - u_2$. Then from (6) and (7) it follows that

$$\left|(A(u_1) - A(u_2), w)\right| \leq \left(1 - \frac{\alpha}{\beta}\right) \|w\|^2.$$

Consequently,

$$\|A(u_1) - A(u_2)\| \leq \left(1 - \frac{\alpha}{\beta}\right) \|u_1 - u_2\|.$$

Thus, the following is true.

Theorem 1. *If the coefficients of equation (1) satisfy inequalities (6), then problem (1), (4) has a unique generalized solution u, to which the following process of successive approximations converges in the metric of* $W_2^{(1)}$*:*

$$\Delta u_{n+1} = \Delta u_n - \frac{1}{\beta} \left[\sum_{i=1}^{n} \frac{\partial a_i(x; u_n, p_{nj})}{\partial x_i} + a_0(x; u_n, p_{nj}) \right],$$

$$u_{n+1}\big|_\Gamma = 0 \qquad (n = 0, 1, \ldots;\ 0;\ u_0 = 0;\ p_{nj} = \frac{\partial u_n}{\partial x_j}).$$

The process converges at the rate of a geometric progression with ratio $q = 1 - \alpha/\beta$*.*

In proving Theorem 1 we used the method applied by I. I. Vorovich and Yu. P. Krasovskii (^{1}) to prove the convergence of the process of elastic solutions in elastoplastic problems. For linear problems an analogous treatment is contained in the paper of L. V. Kantorovich (^{2}).

Remark. Let us note that Theorem 1 loses its force if conditions (2) are not fulfilled.

Indeed, consider the equation

$$\frac{\partial}{\partial x}\left\{\left[\mu\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}+2b\right]\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\left[\mu\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}+2b\right]\frac{\partial u}{\partial y}\right\}+2a=0$$

$$(a, b, \mu = \text{const} > 0)$$

inside the unit disk with center at the origin. We shall seek a solution of this equation that vanishes on the boundary of the disk. It is easy to see that the ellipticity condition (3) is fulfilled for $\alpha = 2b$ ($\xi_0 = 0$), and that this equation has, for all positive a, b, μ, the unique solution

$$u = \frac{b}{\mu}(r-1) - \frac{2}{3a\mu^2}\left[(b^2 + a\mu r)^{3/2} - (b^2 + a\mu)^{3/2}\right],$$

where $r = \sqrt{x^2 + y^2}$. However, as was shown by us in the paper (^{3}), the above process of successive approximations for this equation will diverge for sufficiently large μ.

II. Let us now consider the second boundary condition

$$\sum_{i=1}^{n} a_i \cos(\nu, x_i)\bigg|_{\Gamma} = 0, \tag{8}$$

where ν is the direction of the outward normal to the surface Γ. Let a_0 satisfy the condition

$$\int_{\Omega} a_0 \, dx = 0$$

and let inequalities (6) be fulfilled. Then the solution $u \in W_2^{(1)}(\Omega)$ is still determined by equality (5), which must hold for any function $v \in W_2^{(1)}(\Omega)$. If, however, the functions a_i ($i = 0, 1, \dots, n$) do not depend on u, and inequalities (6) are fulfilled only for $\xi_0 = 0$, then the functions u and v must satisfy the additional condition

$$\int_{\Omega} u \, dx = 0.$$

It is easy to prove that a theorem analogous to Theorem 1 is valid.

The third boundary-value problem with the condition

$$\sum_{i=1}^{n} a_i \cos(\nu, x_i) + \sigma u \Big|_{\Gamma} = 0, \tag{9}$$

where $\sigma(x) \geq \sigma_0 = \text{const}$, is treated in the same way as the first boundary-value problem.

By the same method one can also consider nonhomogeneous boundary conditions; here it suffices to assume that the right-hand sides of the corresponding boundary equalities are traces of functions from $W_2^{(1)}(\Omega)$.

It is easy to see that this same method makes it possible to consider the first boundary-value problem for the equation

$$(-1)^m \sum_{i_1+\cdots+i_n=m} \frac{\partial^m}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \left[a_{i_1,\ldots,i_n}(x; u, \ldots, p_{j_1,\ldots,j_n})\right] +$$

$$+ a_0(x; u, \ldots, p_{j_1,\ldots,j_n}) = 0,$$

where

$$p_{j_1,\ldots,j_n} = \partial^{|j|} u / \partial x_1^{j_1} \cdots \partial x_n^{j_n} \quad (|j| = j_1 + \cdots + j_n \leq m)\,.$$

Here it is assumed that the result of substituting functions from $W_2^{(m)}(\Omega)$ into the coefficients a_i gives functions from $L_2(\Omega)$, and that inequalities corresponding to inequalities (6) hold.

Received
9 X 1961

References

[1] I. I. Vorovich, Yu. P. Krasovskii, DAN, **126**, No. 4, 740 (1959).
[2] L. V. Kantorovich, UMN, **3**, issue 6, 89 (1948).
[3] A. I. Koshelev, Vestn. LGU, issue 4, 20 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.