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Abstract

Full Text

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GEOPHYSICS

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A UNIFIED CHARACTERISTIC OF TURBULENT VISCOSITY FOR SEA WAVES AND CURRENTS

At one time S. V. Dobroklonskii obtained ⁽¹⁾ an expression for the coefficient of turbulent viscosity under wave motion, using Prandtl's and Kármán's concepts of mixing lengths in turbulent motion of a fluid ⁽²⁾. An analogous problem was solved by O. D. Shebalin for a sharply defined shallow sea or lake ⁽³⁾. The relation between the coefficient of turbulent kinematic viscosity ν , the wave height h , and the period T , according to Dobroklonskii, may be written, neglecting small quantities, as

$$\nu = \frac{\pi k^2 h^2}{18 T}. \quad (1)$$

The dimensionless coefficient k entered here from Kármán's relation

$$\nu = k^2 \left| \frac{(dU/dz)^3}{(d^2U/dz^2)^2} \right|, \quad (2)$$

in which dU/dz and d^2U/dz^2 are averaged over the wave period.

Formula (1) proved very useful for constructing a physical theory of wind waves, making it possible to construct the equation of the energy balance of waves without any arbitrary assumptions. Analysis of the limitingly large wind waves in the open ocean that are possible at a given wind speed revealed that the numerical value of k under natural conditions differs considerably from those obtained by Kármán in experiments with turbulent flows in pipes: turbulent viscosity in pipes was characterized by values of k from 0.36 to 0.40, whereas studies of ocean waves gave, in (1), $k = 0.1$. Such a discrepancy seemed paradoxical, since Kármán's calculations may quite legitimately be extended to various kinds of relative motion of fluid masses and, it would seem, k should be

a universal constant. Doubts arose as to the correctness of S. V. Dobroklonskii's accounting for turbulent viscosity in studies of wind waves in the ocean.

The doubts began to dissipate, and confidence in formula (1) strengthened, when S. G. Boguslavskii, studying turbulent exchange in ocean currents and taking into account the difference between the exchange coefficient and the turbulent-friction coefficient, came to the conclusion that Kármán's relation (2), as applied to currents, should contain a value of k close to 0.1, and not Kármán's figures. S. G. Boguslavskii quite justifiably asserts that turbulization under any natural conditions of the open ocean—both in waves and in currents—must differ greatly from that which Kármán studied.

At present, the ratio ν/A of the coefficient of turbulent kinematic viscosity to the coefficient of exchange of heat, salts, and gases is still not sufficiently well known. Therefore, in the present work we have attempted to determine the numerical value of k from formula (2) as applied to sea currents, using only the equations of hydrodynamics and several empirical relations that have received very solid verification.

Let us write, in complex form, the equations of Ekman drift currents:

$$\frac{d^2}{dz^2}(u + iv) = 2ia^2(u + iv). \quad (3)$$

Here u and v are the components of the current velocity at depth z ; through a^2 , as

usually denoted by the expression $\bar{\omega}/\nu$, in which $\bar{\omega} = \omega \sin \varphi$, ω is the angular velocity of the Earth's rotation, and φ is the latitude of the place. Let us agree to measure the angle of rotation of the vector $u + iv$ (at different depths z) from the direction of the velocity vector U_0 of the surface water particles. Then the integral of equation (3) will be represented in the simplest form:

$$u + iv = U_0 e^{-(1+i)az}. \quad (4)$$

Hence one can find expressions for the derivatives entering into (2):

$$\begin{aligned} \frac{d}{dz}(u + iv) &= -a(1 + i)U_0 e^{-(1+i)az}, \\ \frac{d^2}{dz^2}(u + iv) &= a^2(1 + i)^2 U_0 e^{-(1+i)az}. \end{aligned} \quad (5)$$

Substitute (5) into (2). For $z = 0$, for the surface layer we obtain:

$$\nu = k^2 \left| \frac{U_0}{a(1 + i)} \right| = k^2 \frac{U_0}{a\sqrt{2}}. \quad (6)$$

This relation makes it possible to calculate k in two different ways.

1. The velocity U_0 of the surface water particles is related to the wind velocity V by the well-tested empirical relation

$$U_0 = \frac{NV}{\sqrt{\sin \varphi}}. \quad (7)$$

On the basis of (6), (7) one can write, recalling the expression for a^2 :

$$\nu = k^2 \frac{NV}{\sqrt{2\omega}} \frac{\sqrt{\nu}}{\sin \varphi}.$$

Hence follows the first relation suitable for calculating k :

$$k = \left(2\omega \frac{\nu}{V^2}\right)^{1/4} \left(\frac{\sin \varphi}{N}\right)^{1/2}. \quad (8)$$

Here the doubled angular velocity of the Earth's rotation is $2\omega = 1.45 \cdot 10^{-4}$.

On the basis of the studies of N. N. Struiskii, E. Palmén, and G. Thorade,

$$N = 1.27 \cdot 10^{-2}.$$

On the basis of the works of V. Ekman and V. Schmidt, $\nu/V^2 = 4.3 \cdot 10^{-4}$, provided that ν and V are expressed in CGS units. Since the determinations of N were made at middle latitudes, let us set $\varphi = 45^\circ$. Then by formula (8) it turns out that

$$k = 0.117.$$

2. The velocity U_0 can also be related to the tangential force with which the wind acts on a unit surface of the sea:

$$U_0 = \frac{T}{a\delta\nu\sqrt{2}}. \quad (9)$$

On the other hand, the force T itself is expressed by the known relation through the wind velocity V and the air density δ_a :

$$T = \kappa\delta_a^2 V. \quad (10)$$

This relation is sufficiently reliable for $V > 600$ cm/sec.

Substitute expressions (9), (10) into (6).

$$k = \left(\frac{2\omega}{\kappa} \frac{\delta}{\delta_a} \frac{\nu}{V^2} \sin \varphi \right)^{1/2}. \quad (11)$$

Here δ is the density of water; the remaining notation is as before. We take $\delta/\delta_0 = 800$. On the basis of the data of A. Defant, let $\kappa = 2.9 \cdot 10^{-3}$; we retain the remaining numerical values the same as in the preceding

calculations. Then by the second method, from (11), we find

$$k = 0.109.$$

And the first number found, and especially the second, are very close to the one that characterizes turbulent viscosity under wave motion ⁽⁴⁾, $k = 0.1$.

Thus, there is no break in the logical chain that served for studies of turbulent viscosity under completely different conditions, but necessarily under ocean conditions. The conclusion of S. G. Boguslavskii on the unity of the characteristic constant k under natural conditions, and on the unsuitability of the values once obtained by Karman in pipes for describing turbulent processes in the ocean, has been fully confirmed.

It should be recalled that the values of k themselves, obtained for turbulent flows in pipes, differed from one another by 11%. Consequently, one may justifiably neglect the 9% difference between the last figure obtained for currents in the ocean and the figure 0.1 previously calculated by us for waves ⁽⁴⁾.

Now let us return to the Prandtl–Karman representation of the mixing path l and see how this segment is expressed through other, generally known geometric parameters.

Writing the usual expression for l ,

$$l = k \left| \frac{(dU/dz)}{(d^2U/dz^2)} \right|, \quad (12)$$

we substitute into it the derivatives of the vector $u + iv$ appearing in (5). Then it is found that in the upper layer of the drift current

$$l_\tau = \left| \frac{k}{a(1+i)} \right| = \frac{k}{a\sqrt{2}}. \quad (13)$$

Here a can be expressed through the so-called friction depth D , after which a simple and clear geometric relation arises:

$$l_\tau = \frac{kD}{\pi\sqrt{2}} \simeq \frac{D}{45}. \quad (14)$$

One can also express a through the quantities with which we had to deal in deriving (8) and (11), and, ultimately, through the wind speed:

$$l_\tau = k\sqrt{\frac{\nu}{2\bar{\omega}}} = \frac{1.72k}{\sqrt{\sin\varphi}}V \simeq 0.2V. \quad (15)$$

Now let us use formula (12) to determine the length of the mixing path l_v under wave motion. Discarding small terms in the formulas of S. V. Dobroklonskii, on the basis of (9) from the article ⁽¹⁾ we write:

$$\left(\frac{dU}{dz}\right)_0 \simeq \frac{2\pi^3}{T} \left(\frac{h}{\lambda}\right)^2 \quad (16)$$

and on the basis of (10) from the same article:

$$\left(\frac{d^2U}{dz^2}\right)_0 \simeq \frac{12\pi^4}{\lambda T} \left(\frac{h}{\lambda}\right)^2. \quad (17)$$

Substitution of (16), (17) into (12) gives:

$$l_v = \frac{k\lambda}{6\pi} \simeq \frac{\lambda}{188}. \quad (18)$$

On the basis of formulas ⁽⁴⁾, let us express the length of established waves through the speed of the wind that created them. Let us note that the structure of formulas (14), (15) allows l_τ and V to be expressed not only in CGS units: l_τ can be expressed in meters if V is expressed in meters per second. We shall agree to use precisely these units here. Then, in accordance with (5), the length λ of established waves is related to the wind speed by the formula

$$\lambda = 23 \cdot 0.0186V^2 = 0.428V^2. \quad (19)$$

Substituting this expression into (18), we obtain:

$$l_v = 0.00228V^2. \quad (20)$$

At high wind speeds, waves become established slowly, and the regime of the drift current becomes established still more slowly. Therefore, the figures given in Table 1 are purely illustrative material. Here, for established conditions, the values of l_τ and l_v , calculated from (15), (20), the values of ν_τ according to Ekman-Schmidt, and the values of ν_v , calculated from (1) (for $k = 0.1$), are presented. The heights h and periods T of the established waves were determined from (4) for the corresponding wind speeds.

Table 1

V , m/sec	10	15	20	25	30	35
l_τ , m	2	3	4	5	6	7
l_v , m	0.23	0.52	0.91	1.4	2.0	2.8
ν_τ , cm ² /sec	430	970	1720	2700	3870	5270
ν_v , cm ² /sec	1.2	3.9	9.3	18.1	31.3	49.8

Table 1 shows how, with a single characteristic number k , the turbulent phenomena in waves and in flows of drift currents differ sharply from one another. If the development of waves and the development of drift flows were governed by molecular viscosity, then, of course, the coefficient of viscosity would remain the same in both cases: dependent only on intermolecular forces. In contrast, the formal coefficient of turbulent kinematic viscosity describes not the properties of the water itself, but the features of the kinematic conditions (2) in different cases of relative motion of water masses. Similarly, the mixing length is determined by the kinematic conditions (12). It is precisely for this reason that l_τ , l_v differ from one another, and ν_τ , ν_v differ from one another especially sharply.

All our derivations were carried out under the simplified condition widely used in the dynamics of currents: $\nu = \text{const}$. In nature, the phenomena are complicated by the nonconstancy of ν , by its decrease approximately according to the same law by which the moduli of current velocities decrease at depth. To take this circumstance into account, we shall use the integral of Ekman's more complicated equations, which he obtained long ago⁵ and which is rarely cited.

In complex form, instead of (4) we write:

$$u + iv = U_0 \left(1 - 0.8 \frac{z}{D_1}\right)^3 e^{3.46i \ln(1 - 0.8z/D_1)}. \quad (21)$$

Repeating all our derivations, we now obtain, instead of (6):

$$\nu = 0.364k^2 U_0 D_1, \quad (22)$$

where the new value of the friction depth D_1 , according to⁵, can be expressed through the wind speed:

$$D_1 = \frac{7.5V}{\sqrt{\sin \varphi}}. \quad (23)$$

Again using reliable empirical data for N and for ν/V^2 , we arrive at the numerical value

$$k \simeq 0.102,$$

which is still closer to the figure obtained by us for ocean waves.

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