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Abstract

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MATHEMATICS

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ON SELF-ADJOINT OPERATORS IN THE ORTHOGONAL SUM OF HILBERT SPACES

(Presented by Academician A. N. Kolmogorov 12 I 1962)

Recall that a linear operator A , acting in a Hilbert space H , is called Hermitian if for any elements f, g from its domain of definition D_A , $(Af, g) = (f, Ag)$. If, in addition, the manifold D_A is dense in H , then the operator A is called symmetric. M. A. Naimark^(1,2) described the structure of a self-adjoint operator that acts in the orthogonal sum of two Hilbert spaces and is an extension of a closed symmetric operator acting in one of the spaces. This description is given in terms connected with the Cayley transform. If, instead of a symmetric operator in one of the spaces, a closed Hermitian operator is given, then the description of all possible self-adjoint extensions of such an operator, also based on the Cayley transform, is given by results of M. A. Krasnoselskii^(3,4). In the present paper we study the structure of a self-adjoint operator in the orthogonal sum of two Hilbert spaces under the assumption that it is an extension of a closed Hermitian operator with finite defect numbers, acting in one of the spaces. The structure of such a self-adjoint operator is described in terms of the so-called boundary spaces and boundary operators.

1. A linear operator A , acting in the orthogonal sum $H = H_1 \oplus H_2$ of Hilbert spaces H_1 and H_2 , will be called a simple coupling of linear operators B_1, B_2 , acting respectively in H_1 and H_2 , if $P_k D_A = D_{B_k}$ and for any $f \in D_A$, $P_k A f = B_k P_k f$ ($k = 1, 2$), where P_k is the projection operator in H onto H_k .

For the study of simple couplings that are Hermitian operators, we shall use the concepts of a boundary space and a boundary operator introduced in⁽⁵⁾. Recall that a linear space \mathcal{L} with a nondegenerate scalar product $[\varphi, \psi]$, in general indefinite, is called a boundary space of a linear operator B , acting in a Hilbert space, if there exists a linear operator Γ , mapping D_B onto \mathcal{L} , such that for any

$$f, g \in D_B \quad [\Gamma f, \Gamma g] = \frac{1}{i} [(Bf, g) - (f, Bg)];$$

the operator Γ is then called a boundary operator.

Lemma. Let the linear operators B_1 in H_1 and $-B_2$ in H_2 have a common boundary space \mathcal{L} , and let $\Gamma_{B_1}, \Gamma_{-B_2}$ be some of their boundary operators with common range \mathcal{L} . Then the simple coupling A of the operators B_1, B_2 , defined on the manifold D_A of all possible elements $\{f_1, f_2\} \in H_1 \oplus H_2$ such that $f_1 \in D_{B_1}, f_2 \in D_{B_2}, \Gamma_{B_1} f_1 = \Gamma_{-B_2} f_2$, is a Hermitian operator. Conversely, if the simple coupling A of the operators B_1, B_2 is a Hermitian operator, then $B_1, -B_2$ have a common boundary space, and the domain of definition D_A of the operator A is specified in the manner indicated above by means of certain boundary operators $\Gamma_{B_1}, \Gamma_{-B_2}$.

We shall agree to say that in the case described in the lemma, the simple coupling A of the operators B_1, B_2 is defined by the boundary operators $\Gamma_{B_1}, \Gamma_{-B_2}$.

If A is any linear operator in a Hilbert space H , and H_1 is a subspace in H , then by a part of the operator A generated in H_1 we shall mean the operator $A_1 \subset A$, defined on the manifold D_{A_1} of all $f \in D_A \cap H_1$ for which $Af \in H_1$. In the case when A is a closed operator, A_1 is also closed. If A is a self-adjoint operator in $H = H_1 \oplus H_2$, then its parts A_1 and A_2 , generated respectively in H_1 and H_2 , are closed Hermitian operators, and the defect indices of the operators A_1 and $-A_2$ are the same.

Theorem 1. Let A_1, A_2 be closed symmetric operators in H_1, H_2 such that the defect numbers of the operators $A_1, -A_2$, respectively, are equal and finite. Then the operators $A_1^*, -A_2^*$ possess a common boundary space u , whatever the boundary operators $\Gamma_{A_1^*}, \Gamma_{-A_2^*}$ with common range defining it may be; the simple coupling A of the operators A_1^*, A_2^* is a self-adjoint operator in $H = H_1 \oplus H_2$. Moreover, the parts of the operator A generated in H_1, H_2 coincide respectively with A_1, A_2 .

Theorem 2. Let A be a self-adjoint operator in $H = H_1 \oplus H_2$. If its parts A_1, A_2 , generated respectively in H_1, H_2 , are symmetric operators with finite defect numbers, then the operator A is a simple coupling of the operators A_1^*, A_2^* , defined by some boundary operators $\Gamma_{A_1^*}, \Gamma_{-A_2^*}$.

2. Let A be a closed Hermitian operator in H . A closed linear operator B in H , which is an extension of the operator A and has domain D_B dense in H , will be called a special extension of the operator A if the linear manifold G_B of all elements $f \in D_B \cap D_{B^*}$ for which $Bf = B^*f$ coincides with D_A . From this definition it follows that, simultaneously with B , B^* is also a special extension of the operator A .

We note the particular case when the operator A is symmetric. In order that a closed operator B be a special extension of a closed symmetric operator A , it is necessary and sufficient that the conditions hold: 1) $A \subset B \subset B^*$; 2)

$D_B \cap D_{B^*} = D_A$. In particular, A and A^* are special extensions of the operator A .

Suppose that the closed Hermitian operators A_1 and $-A_2$, acting respectively in H_1 and H_2 , have the same defect indices. Let B_1 in H_1 and B_2 in H_2 be special extensions of the operators A_1 and A_2 such that the operators B_1 and $-B_2$ have a common boundary space \mathcal{L} , while $-B_1^*$ and B_2^* have a common boundary space \mathcal{L}' .^{*} Consider any simple couplings C and C' of the operators B_1, B_2 and B_1^*, B_2^* , defined respectively by the boundary operators $\Gamma_{B_1}, \Gamma_{-B_2}$ and $\Gamma_{-B_1^*}, \Gamma_{B_2^*}$. The operators C and C' coincide on the manifold $D_C \cap D_{C'}$ if and only if $D_C \cap D_{C'} = D_{A_1} \oplus D_{A_2}$, and this is equivalent to the following condition imposed on the boundary operators: if, for some elements $f_1 \in D_{B_1} \cap D_{B_1^*}$ and $f_2 \in D_{B_2} \cap D_{B_2^*}$, $\Gamma_{B_1} f_1 = \Gamma_{-B_2} f_2$ and $\Gamma_{-B_1^*} f_1 = \Gamma_{B_2^*} f_2$, then $f_1 \in D_{A_1}$ and $f_2 \in D_{A_2}$. Assuming this condition to be fulfilled,^{**} we shall call the minimal common linear extension A

^{*} In that particular case when the defect numbers of the operators A_1 and A_2 are finite, it is enough to assume that the common boundary space \mathcal{L} is possessed by the operators B_1 and B_2 ; hence it already follows that the operators $-B_1^*$ and B_2^* also possess a common boundary space \mathcal{L}' .

^{**} If at least one of the operators A_1 or A_2 is symmetric, then the indicated condition is fulfilled for any boundary operators $\Gamma_{B_1}, \Gamma_{-B_2}, \Gamma_{-B_1^*}, \Gamma_{B_2^*}$.

operators C and C' , by the double coupling of the operators B_1, B_2, B_1^*, B_2^* , defined by the boundary operators $\Gamma_{B_1}, \Gamma_{-B_2}, \Gamma_{-B_1^*}, \Gamma_{B_2^*}$; the operators B_1, B_2, B_1^*, B_2^* shall conditionally be called the components of the double coupling A .

Theorem 3. Let the operators B_1 in H_1 and B_2 in H_2 be special extensions of the closed Hermitian operators A_1, A_2 such that $B_1, -B_2$, and also $-B_1^*, B_2^*$, have common boundary spaces. Then the double coupling A of the operators B_1, B_2, B_1^*, B_2^* , defined by the boundary operators $\Gamma_{B_1}, \Gamma_{-B_2}, \Gamma_{-B_1^*}, \Gamma_{B_2^*}$, is a Hermitian operator in $H = H_1 \oplus H_2$, and its parts generated in H_1, H_2 coincide respectively with the operators A_1, A_2 . If the deficiency numbers of the operators A_1, A_2 are finite, then A is a self-adjoint operator.

Theorem 4. Let A be a self-adjoint operator in $H = H_1 \oplus H_2$ such that its parts A_1, A_2 , generated respectively in H_1, H_2 , have finite deficiency numbers. Then there exist special extensions B_1, B_2 of the operators A_1, A_2 , respectively, such that A is the double coupling of the operators B_1, B_2, B_1^*, B_2^* , defined by certain boundary operators $\Gamma_{B_1}, \Gamma_{-B_2}, \Gamma_{-B_1^*}, \Gamma_{B_2^*}$.

We note that, under the conditions of Theorem 4, the operators B_1 in H_1 and B_2 in H_2 , which are components of the double coupling equal to the operator A , are in general not uniquely determined by this operator. However, as is easily seen, from the operators A and B_1 the operator B_2 is reconstructed uniquely;

its domain D_{B_2} is the manifold of all elements $f_2 \in H_2$ for each of which there exists an element $f_1 \in D_{B_1}$ such that $f_1 + f_2 \in D_A$ and $P_1 A(f_1 + f_2) = B_1 f_1$; in this case $B_2 f_2 = P_2 A(f_1 + f_2)$; P_1 and P_2 denote, as before, the projection operators in H onto H_1 and H_2 , respectively.

Theorem 5. Let the operators A in $H = H_1 \oplus H_2$, A_1 in H_1 , and A_2 in H_2 be the same as in Theorem 4. A special extension B_1 of the operator A_1 is a component of some double coupling equal to the operator A if and only if the following conditions are satisfied:

- 1) $B_1 f_1 = P_1 A f_1$ only when $f_1 \in D_{A_1}$; 2) $B_1^* g_1 = P_1 A g_1$ only when $g_1 \in D_{A_1}$.

If the operator A_2 is symmetric, then every special extension B_1 of the operator A_1 satisfies these conditions*.

3. Let us illustrate the preceding constructions on the simplest examples.

- 1) Let $H = \mathcal{L}^2(0, 2)$, $H_1 = \mathcal{L}^2(0, 1)$, $H_2 = \mathcal{L}^2(1, 2)$; let A be the differentiation operator in $\mathcal{L}^2(0, 2)$ generated by the operation $i \frac{d}{dx}$ and the boundary condition $y(0) = y(2)$. A_1 and A_2 are symmetric differentiation operators in $\mathcal{L}^2(0, 1)$ and $\mathcal{L}^2(1, 2)$, defined respectively by the boundary conditions: $y(0) = y(1) = 0$, $y(1) = y(2) = 0$.

A common boundary space of the operators A_1^* and $-A_2^*$ is a two-dimensional space with an indefinite metric. The operator A is the simple coupling of the operators A_1^* and A_2^* , defined by the boundary operators $\Gamma_{A_1^*}$ and $\Gamma_{-A_2^*}$, which are given by the formulas:

$$\Gamma_{A_1^*} f_1 = \{f_1(1), f_1(0)\} \quad (f_1 \in D_{A_1^*}); \quad \Gamma_{-A_2^*} f_2 = \{f_2(1), f_2(2)\} \quad (f_2 \in D_{A_2^*}).$$

- 2) Let H, H_1, H_2 and the operator A be the same as in 1). Consequently, the operators A_1, A_2 also remain the same. Let B_1 be an operator, ge-

* We note that among the special extensions B_1 of the operator A_1 satisfying the conditions of Theorem 5 there are some that possess regular points in the complex plane. This fact is essential for applications of the results presented to the theory of characteristic functions of linear operators, on which we do not dwell here.

generated in $\mathcal{L}^2(0, 1)$ by the operation $i \frac{d}{dx}$ and the boundary condition $y(0) = 0$. B_1 is a special extension of the operator A_1 . The corresponding special extension B_2 of the operator A_2 is generated in the space $\mathcal{L}^2(1, 2)$ by the same operation $i \frac{d}{dx}$ and the boundary condition $y(2) = 0$. The boundary space of each of the operators $B_1, -B_2, -B_1^*, B_2^*$ is a one-dimensional unitary space. The operator

A is a double coupling of the operators B_1, B_2, B_1^*, B_2^* , defined by the boundary operators $\Gamma_{B_1}, \Gamma_{-B_2}, \Gamma_{-B_1^*}, \Gamma_{B_2^*}$, which are given by the formulas:

$$\Gamma_{B_1} f_1 = f_1(1) \quad (f_1 \in D_{B_1}); \quad \Gamma_{-B_2} f_2 = f_2(1) \quad (f_2 \in D_{B_2});$$

$$\Gamma_{-B_1^*} g_1 = g_1(0) \quad (g_1 \in D_{B_1^*}); \quad \Gamma_{B_2^*} g_2 = g_2(2) \quad (g_2 \in D_{B_2^*}).$$

- 3) Let the space H be countably dimensional and let $e_0, e_1, e_2, \dots, e_k, \dots$ be an orthonormal basis in H . Consider the linear symmetric operator A' in H , defined on all possible finite linear combinations of the elements of the basis according to the formulas:

$$A'e_0 = a_0 e_0 + b_0 e_1, \quad A'e_k = b_{k-1} e_{k-1} + a_k e_k + b_{k+1} e_{k+1} \quad (k = 1, 2, \dots),$$

where a_k and b_k are real numbers, with $b_k \neq 0$ ($k = 0, 1, 2, \dots$). Denote the closure of the operator A' by A . As is known, the operator A is either self-adjoint or symmetric with defect index (1.1). Suppose that the first case occurs. Fixing some natural number m , denote by H_1 the linear span of the system of elements e_0, e_1, \dots, e_m , and put $H_2 = H \ominus H_1$. It is easy to see that the parts A_1 and A_2 of the operator A , generated respectively in H_1 and H_2 , are Hermitian operators with defect index (1.1) and have nondense domains of definition in these subspaces. Define the operator B_1 in H_1 as a linear extension of the operator A_1 , setting

$$B_1 e_m = b_{m-1} e_{m-1} + (a_m + i b_m) e_m.$$

The operator B_1 is a special extension of the operator A_1 , satisfying all the conditions of Theorem 5. Therefore A can be represented in the form of a double coupling, one of whose components is B_1 . The second component B_2 is now determined uniquely: B_2 is the linear extension of the operator A_2 , defined on the linear span of the manifold D_{A_2} and the element e_{m+1} by the formula

$$B_2 e_{m+1} = (a_{m+1} - i b_m) e_{m+1} + b_{m+1} e_{m+2}.$$

The operators $B_1, -B_2, -B_1^*, B_2^*$ have a common one-dimensional boundary space. Define the boundary operators as follows:

$$\Gamma_{B_1} f_1 = \Gamma_{-B_1^*} f_1 = (f_1, e_m) \quad (f_1 \in H_1);$$

$$i\Gamma_{-B_2} f_2 = -i\Gamma_{B_2^*} f_2 = (f_2, e_{m+1}) \quad (f_2 \in D_{B_2} = D_{B_2^*}).$$

It is easy to verify that A is a double coupling of the operators B_1, B_2, B_1^*, B_2^* , defined by the indicated boundary operators.

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Note: Figure translations are in progress. See original paper for figures.

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