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Abstract

Full Text

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CUBATURE FORMULAS FOR COMPUTING INTEGRALS OVER A HYPERSPHERE

(Presented by Academician V. I. Smirnov on 2 VI 1962)

Cubature formulas for computing an n -fold integral of a function $f(x)$ over a domain D in n -dimensional space have the form

$$\int_D f(x) dx \simeq \sum_{j=1}^N C_j f(x^{(j)}), \tag{1}$$

where $x = (x_1, x_2, \dots, x_n)$, $dx = dx_1 dx_2 \dots dx_n$, $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)})$ are the nodes and C_j are the coefficients of the formula, and they are a natural generalization of quadrature formulas for computing single integrals. Just as natural, at least from the theoretical point of view, one may regard the following generalization of one-dimensional quadrature formulas:

$$\int_D f(x) dx \simeq \sum_{j=1}^p \int_{S_j} p_j(x) f(x) dS_j. \tag{2}$$

Here S_j are $(n - 1)$ -dimensional surfaces situated in D ("nodes"); $p_j(x)$ are functions defined on S_j ("coefficients"), and the integrals on the right-hand side are surface integrals.

In the present note we indicate cubature formulas of the highest algebraic degree of precision of the form (2) for the case in which the region of integration is the n -dimensional ball of unit radius. In what follows D denotes the ball

$$x_1^2 + x_2^2 + \dots + x_n^2 \leq 1.$$

With the help of formulas of the form (2), ordinary cubature formulas (1) are constructed. In particular, for $n = 2$ and $n = 3$ we obtain the known cubature formulas for the disk and the ball (1-3).

In view of the symmetry, as S_j we take the surface of an n -dimensional ball with center at the origin and radius r_j , and as $p_j(x)$ a constant A_j . Formula (2) will be written as follows:

$$\sum_{j=1}^p A_j r_j^{2\nu+n-1} = \frac{1}{2\nu+n}, \quad \nu = 0, 1, 2, \dots, 2p-1. \quad (7)$$

It is not difficult to verify that for even n the solution of the system (7) is given by the numbers r_j and A_j determined by the equalities

$$t_j = r_j^2, \quad D_j = 2A_j r_j^{n-1} \quad (j = 1, 2, \dots, p).$$

Here t_j and D_j are the nodes and coefficients of a Gaussian-type quadrature formula for the interval $[0, 1]$ and weight $t^{n/2-1}$:

$$\int_0^1 t^{n/2-1} \varphi(t) dt \cong \sum_{j=1}^p D_j \varphi(t_j). \quad (8)$$

For odd n , the r_j are the positive nodes of a Gaussian-type quadrature formula with $2p$ nodes for the interval $[-1, 1]$ and weight r^{n-1} :

$$\int_{-1}^1 r^{n-1} \varphi(r) dr \cong \sum_{j=-p}^p {}' E_j \varphi(r_j), \quad (9)$$

and the A_j are determined by the equalities $E_j = A_j r_j^{n-1}$, where E_j are the coefficients of formula (9). The prime on the summation sign in (9) means that the term corresponding to $j = 0$ is omitted.

From the method of constructing formula (3) it is clear that it becomes an exact equality when $f(x)$ is a polynomial of degree $4p - 1$.

We construct cubature formulas

$$\int_D f(x) dx \cong Bf(O) + \sum_{j=1}^p B_j \int_{S_j} f(x) dS_j, \quad (10)$$

which differ from (3) in that, to the terms of the kind indicated in (3), there has been added

the term $Bf(O)$, $O = (0, 0, \dots, 0)$. As above, we obtain that the unknowns r_j, B_j, B must satisfy the system of equations

$$\frac{\Gamma(n/2)}{2\pi^{n/2}} B + \sum_{j=1}^p B_j r_j^{n-1} = \frac{1}{n}, \quad \sum_{j=1}^p B_j r_j^{2\nu+n-1} = \frac{1}{2\nu+1}. \quad (11)$$

Let n be even. The solution of system (11) is given by the numbers r_j, B_j, B determined by the equalities

$$\tau_j = r_j^2, \quad G_j = 2B_j r_j^{n-1}, \quad G = \frac{\Gamma(n/2)}{\pi^{n/2}} B,$$

where τ_j, G_j , and G are the nodes and coefficients of the A. A. Markov quadrature formula with weight $t^{n/2-1}$ on the interval $[0, 1]$ and with one fixed node $t = 0$ (see (4)):

$$\int_0^1 t^{n/2-1} \varphi(t) dt \cong G\varphi(0) + \sum_{j=1}^p G_j \varphi(\tau_j). \quad (12)$$

For odd n , the r_j are the positive nodes of a Gaussian quadrature formula with weight r^{n-1} on the interval $[-1, 1]$ and with $2p + 1$ nodes:

$$\int_{-1}^1 r^{n-1} \varphi(r) dr \cong \sum_{j=-p}^p H_j \varphi(r_j), \quad (13)$$

and the coefficients B and B_j are determined by the equalities

$$\frac{\Gamma(n/2)}{\pi^{n/2}} B = H_0, \quad B_j r_j^{n-1} = H_j, \quad j = 1, 2, \dots, p,$$

where H_j are the coefficients of formula (13).

Formula (10), as follows from the method of its construction, is exact when $f(x)$ is a polynomial of degree $4p + 1$.

Using formulas (3) and (10), we construct cubature formulas of the form (1). We begin with formula (3). We replace the surface integral in (3) by an ordinary one, using the spherical coordinates (5):

$$\begin{aligned} I_j &= \int_{S_j} f(x) dS_j = \\ &= r_j^{n-1} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} f(x) \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2} d\varphi_1 d\varphi_2 \cdots d\varphi_{n-1}. \end{aligned} \quad (14)$$

If $f(x)$ is a polynomial of degree $4p - 1$, then the integrand in (14) is a trigonometric polynomial of degree $4p - 1$ with respect to the variable φ_{n-1} . Performing the integration with respect to φ_{n-1} , we obtain an $(n - 2)$ -fold integral in which

the integrand is the product of $\sin \varphi_{n-2}$ and a trigonometric polynomial of degree $4p - 1$, which, as a function of φ_{n-2} , depends only on $\cos \varphi_{n-2}$. Indeed, odd powers of $\sin \varphi_{n-2}$ can arise only from monomials containing the factor $x_{n-1}^\alpha x_n^\beta$, where $\alpha + \beta$ is odd. But if $\alpha + \beta$ is odd, then the integral of such a monomial with respect to φ_{n-1} is equal to zero. Similarly, as a result of integration with respect to $\varphi_{n-1}, \varphi_{n-2}, \dots, \varphi_{n-k-1}$ ($k = 0, 1, \dots, n-3$), we obtain an $(n - k - 2)$ -fold integral in which the integrand is the product of $\sin^{k+1} \varphi_{n-k-2}$ and a trigonometric polynomial of degree $4p - 1$, which, as a function of φ_{n-k-2} , depends only on $\cos \varphi_{n-k-2}$.

We replace the integral in (14) with respect to the variable φ_{n-1} by a quadrature sum according to the rectangle formula with $4p$ nodes. When replacing the integral with respect to φ_{n-k-2}

($k = 0, 1, 2, \dots, n - 3$), for the quadrature sum we use a formula of Gaussian type with weight $(1 - t^2)^{k/2}$ on the interval $[-1, 1]$ and with $2p$ nodes:

$$\int_{-1}^1 (1 - t^2)^{k/2} \varphi(t) dt \cong \sum_{m=1}^{2p} P_m^{(k)} \varphi(t_m^{(k)}). \quad (15)$$

If we use the notation

$$\begin{aligned} F_j(\varphi_1, \varphi_2, \dots, \varphi_{n-1}) &= \\ &= f(r_j \cos \varphi_1, r_j \sin \varphi_1 \cos \varphi_2, \dots, r_j \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-1}), \end{aligned}$$

we obtain

$$\begin{aligned} I_j &\cong r_j^{n-1} \frac{\pi}{2p} \sum_{i_1=1}^{4p} \sum_{i_2, \dots, i_{n-2}=1}^{2p} P_{i_1}^{(n-3)} P_{i_2}^{(n-4)} \dots P_{i_{n-2}}^{(0)} \times \\ &\times F_j \left(\varphi_1^{(i_1)}, \varphi_2^{(i_2)}, \dots, \varphi_{n-2}^{(i_{n-2})}, \frac{\pi}{2p} i \right), \end{aligned} \quad (16)$$

where

$$\varphi_{n-k-2}^{(m)} = \arccos t_m^{(k)},$$

$t_m^{(k)}$ and $P_m^{(k)}$ are the nodes and coefficients of formula (15). Equality (16) becomes exact when $f(x)$ is a polynomial of degree $4p - 1$.

Comparing (3), (14), and (16), we obtain a cubature formula with $N = (2p)^n$ nodes and exact for any polynomial of degree $4p - 1$. In particular, for $n = 2$

we obtain the formula of L. V. Kantorovich ⁽¹⁾, and for $n = 3$ the formula of V. A. Ditkin ⁽³⁾.

In an analogous way, with the aid of formula (10), we obtain

$$\int_D f(x) dx \cong Bf(0) + \sum_{j=1}^p B_j r_j^{n-1} \frac{\pi}{2p+1} \sum_{i=1}^{4p+2} \sum_{i_1, i_2, \dots, i_{n-2}=1}^{2p+1} Q_{i_1}^{(n-3)} Q_{i_2}^{(n-4)} \dots Q_{i_{n-2}}^{(0)} \times \times F_j \left(\varphi_1^{(i_1)}, \varphi_2^{(i_2)}, \dots, \varphi_{n-2}^{(i_{n-2})}, \frac{\pi}{2p+1} i \right). \quad (17)$$

Here $\varphi_{n-k-2}^{(m)} = \arccos \tau_m^{(k)}$, and $\tau_m^{(k)}$ and $Q_m^{(k)}$ are the nodes and coefficients of the Gaussian-type quadrature formula with weight $(1-t^2)^{k/2}$ on the interval $[-1, 1]$ and with $2p+1$ nodes

$$\int_{-1}^1 (1-t^2)^{k/2} \varphi(t) dt \cong \sum_{m=1}^{2p+1} Q_m^{(k)} \varphi(\tau_m^{(k)}).$$

The cubature formula (17) has $N = 2p(2p+1)^{n-1} + 1$ nodes and becomes an exact equality when $f(x)$ is any polynomial of degree $4p+1$. In particular, for $n = 2$ we obtain the formula of L. A. Lyusternik ⁽²⁾, and for $n = 3$ the formula of V. A. Ditkin ⁽³⁾.

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REFERENCES

1. L. V. Kantorovich, *Proceedings of the Mathematical Institute named after V. A. Steklov, Academy of Sciences of the USSR*, **28**, 3 (1949).
2. L. A. Lyusternik, *DAN*, **62**, No. 4, 449 (1948).
3. V. A. Ditkin, *DAN*, **62**, No. 4, 445 (1948).
4. V. I. Krylov, *Approximate Calculation of Integrals*, 1959.

Note: Figure translations are in progress. See original paper for figures.

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