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Abstract

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THEORY OF DIFFERENTIAL EQUATIONS WITH A SMALL MULTIPLIER AT THE DERIVATIVES IN LINEAR TOPOLOGICAL SPACES

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I. Introduction. In the present paper, the theory of differential equations with small multipliers at the derivatives, developed by A. N. Tikhonov and I. S. Gradshtein (^{1}-^{5}), is generalized to the case of arbitrary linear topological spaces.

II. Definitions. Consider the differential equation

$$\frac{dx}{dt} = f(x, t) \tag{1}$$

in an arbitrary linear topological space L ($x \in L$, t is a real number) (^{6}).

Definition 1. Let a set of equations be given

$$\frac{dx}{dt} = f(x, p), \tag{2}$$

where the parameter $p \in P$ (a set of arbitrary nature). Let $R = \{(x, p)\}$ be some set of singular points of these equations, i.e. $f(R) = 0$. The set of equations (2) is called **uniformly asymptotically stable** (in the positive direction) with respect to R, if for any neighborhoods of zero U and $V \subset U$ there exist a neighborhood of zero $W(U)$ and a real number $\theta(W, V)$, such that for every $(x_0, p_0) \in R$, from the fact that

$$x(0, p_0) - x_0 \in W$$

it follows that

$$\begin{aligned} x(\tau, p_0) - x_0 \in U \quad & \text{for } \tau \geq 0, \\ x(\tau, p_0) - x_0 \in V \quad & \text{for } \tau \geq \theta(W, V). \end{aligned}$$

Definition 2. Let $(x_0, p_0) \in R$. The set $S_{(x_0,p_0)}$ of all points (x, p_0) such that

$$x(\tau) \longrightarrow x_0, \qquad \tau \to +\infty,$$

where $x(\tau)$ is the solution of the initial-value problem

$$\frac{dx}{d\tau} = f(x, p_0), \qquad x(0) = x_0,$$

is called the **set separated by the point** (x_0, p_0). The set

$$S = \bigcup_{(x_0,p_0)\in R} S_{(x_0,p_0)}$$

is called the **envelope of the set** R.

III. Let $L_1, L_2, \dots, L_k$ be arbitrary linear topological spaces. Consider the system of equations

$$\varepsilon^{n_i} \frac{dx_i}{dt} = f_i(x_1, \dots, x_k, t) \quad (i = 1, 2, \dots, k); \tag{3}$$

$$n_1 > n_2 > \dots > n_k \geq 0; \qquad x_i \in L_i \ (i = 1, \dots, k).$$

(The case when among the n_i some are equal to one another is, obviously, easily reduced to this one.)

Theorem.[1] Let $z_0(t) = \{x_{1,0}(t); \dots; x_{k,0}(t)\}$ be a continuous solution of the initial-value problem:

$$0 = f_1(x_1, \dots, x_k, t),$$

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_k, t) \qquad (i = 2, \dots, k),$$

$$x_{i,0}(t_0) = x_{i,0} \qquad (i = 1, \dots, k).$$

2) Let the set of equations

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[1] 1) in the source text.

$$\frac{dx}{d\tau} = f_1(x_1, x_2, \dots, x_k, t), \tag{4}$$

where $(x_2, \dots, x_k, t) = p$ is a parameter, be uniformly asymptotically stable (in the positive direction) with respect to $R = z_0([t, T])$. Let

$$(\bar{x}_1; x_{2,0}; \dots; x_{k,0}; t) \in S_{(x_{1,0}; x_{2,0}; \dots; x_{k,0}; t_0)}$$

(see Definition 2).

3) Suppose that for the equation in $L_1 \times L_2 \times \cdots \times L_k$

$$\frac{dx_1'}{d\tau} = f_1(x_1', x_2', \dots, x_k', t^* + \varepsilon^{n_1}\tau),$$

$$\frac{dx_i'}{d\tau} = \varepsilon^{n_1 - n_i} f_i(x_1', x_2', \dots, x_k', t^* + \varepsilon^{n_1}\tau) \qquad (i = 2, \dots, k), \tag{5}$$

which we write briefly as

$$\frac{dz'}{d\tau} = \varphi(\varepsilon, z', \tau),$$

there is continuous dependence on the parameter: for every neighborhood of zero $U \subset L_1 \times L_2 \times \cdots \times L_k$ and every number $\theta > 0$ there exists $\delta > 0$ such that, for all $t^* \in [t_0, T]$, $\varepsilon < \delta$, $z_0 \in S$ (see Definition 2), the solutions $z(\varepsilon, \tau)$ and $z(0, \tau)$ of the initial-value problems

$$\frac{dz'}{d\tau} = \varphi(\varepsilon, z', \tau), \qquad z'(0) = z_0,$$

$$\frac{dz'}{d\tau} = \varphi(0, z', \tau), \qquad z'(0) = z_0$$

are such that $z'(\varepsilon, \tau) - z'(0, \tau) \in U$ for every $\tau \in [0, \theta]$.

4) Suppose that for system (3), for all initial data $(x_1(t_0), \dots, x_k(t_0)) \in S$, the initial-value problem (Cauchy problem) has a unique solution.

If these conditions are satisfied, then the solution $z(\varepsilon, t)$ of the initial-value problem

$$(x_1(t_0), x_2(t_0), \dots, x_k(t_0)) = (\bar{x}_1, x_{2,0}, \dots, x_{k,0}) \tag{3'}$$

for system (3) is such that

$$z(\varepsilon, t) \xrightarrow[\varepsilon\to+0]{} z(0,t)$$

uniformly on $[t_1, T]$, where t_1 is arbitrary $> t_0$.

Proof. Make the substitution in system $(3')$:

$$t = t^* + \varepsilon^{n_1}\tau, \qquad x_i(t) = x_i(t^* + \varepsilon^{n_1}\tau) = x_i'(\tau)$$

(i.e., $z(t) = z'(\tau)$). For $t^* = t_0$ we obtain system (5) with the initial data

$$x_1'(0) = \bar{x}_1, \qquad x_i'(0) = x_{i,0} \qquad (i = 2, \dots, k). \tag{5'}$$

We make an analogous substitution in $z_0(t)$:

$$z_0(t) = z_0(t^* + \varepsilon^{n_1}\tau) = z_0'(\tau).$$

We shall carry out the proof of the theorem in three stages:

1. **Lemma 1.** Put $t^* = t_0$. Then for every neighborhood of zero

   $$V = V_1 \times \dots \times V_k \subset L_1 \times \dots \times L_k$$

   there exist positive numbers $\theta_1 > 0$ and $\delta_1 > 0$ such that, for every $\varepsilon < \delta_1$, for the solution $z'(\varepsilon, \tau)$ of system $(5')$ one has

$$z'(\varepsilon,\tau) - z_0'(\tau) \in V \quad \text{for } \tau \in [\theta_1, 2\theta_1].$$

Proof. A. From condition 2) we find θ_1 such that

$$x_i'(0,\tau) - x_0 \in \frac{1}{3}V_1 \quad \text{for } \tau \geq \theta_1.$$

Moreover, $x_i'(0,\tau) = x_{i,0}$ $(i = 2, \dots, k)$, and therefore

$$x_i'(0,\tau) - x_{i,0} \in \frac{1}{3}V_i \quad (i = 2, \dots, k),$$

hence

$$z'(0,\tau) - z_0 \in \frac{1}{3}V \quad \text{for } \tau \geq \theta_1.$$

B. From condition 3), for θ_1 we find $\delta_0 > 0$ such that, for every $\varepsilon < \delta_0$,

$$z'(\varepsilon,\tau) - z'(0,\tau) \in \frac{1}{3}V \quad \text{for } \tau \in [0, 2\theta_1].$$

C. From the continuity of $z_0(t)$ (condition 1)) we find $\delta_2 > 0$ such that

$$z_0(t) - z_0 \in \frac{1}{3}V \quad \text{for } t \in [t_0, t_0 + 2\delta_2].$$

Hence, for $\varepsilon < \delta_3 = (\delta_2/\theta_1)^{1/n_1}$,

$$z_0'(\tau) - z_0 \in \frac{1}{3}V \quad \text{for } \tau \in [0, 2\theta_1].$$

D. Choosing $\delta_1 = \min(\delta_0, \delta_3)$ and combining the results of A, B, C, we obtain: for every $\varepsilon < \delta_1$,

$$z'(\varepsilon, \tau) - z_0'(\tau) \in V \quad \text{for } \tau \in [\theta_1, 2\theta_1].$$

2. **Lemma 2.** For every neighborhood of zero

$$U \subset L_1 \times \cdots \times L_k$$

there exist a neighborhood of zero

$$V \subset L_1 \times \cdots \times L_k$$

and numbers $\theta_0 > 0$, $\delta_0 > 0$ such that, if for $t^* \in [t_0, T]$ (see the substitution before Lemma 1)

$$z'(\varepsilon, 0) - z_0'(0) \in V,$$

then for every $\varepsilon < \delta_0$:

1)
$$z'(\varepsilon, \tau) - z_0'(\tau) \in U \quad \text{for } \tau \in [0, \theta_0];$$

2)
$$z'(\varepsilon, \tau) - z_0'(\tau) \in V \quad \text{for } \tau \in [\theta_0, 2\theta_0].$$

Proof. A. From condition 2) we find $V \subset U$ and $\theta_0 > 0$ such that:

1)
$$z'(0, \tau) - z_0 \in \frac{1}{3}U \quad \text{for } \tau \in [0, \theta_0];$$

2)
$$z'(0, \tau) - z_0 \in \frac{1}{3}V \quad \text{for } \tau \in [\theta_0, 2\theta_0].$$

B. From condition 3) we find $\delta_1 > 0$ such that, for $\varepsilon < \delta_1$,

$$z'(\varepsilon,\tau) - z'(0,\tau) \in \frac{1}{3}V \quad \text{for } \tau \in [0, 2\theta_0].$$

C. From the uniform continuity of $z_0(t)$ on the interval $[t_0, T]$, we find, as in point C of the proof of Lemma 1, $\delta_2 > 0$ such that, for $\varepsilon < \delta_2$,

$$z_0'(\tau) - z_0 \in \frac{1}{3}V \quad \text{for } \tau \in [0, 2\theta_0].$$

D. Choosing $\delta_0 = \min(\delta_1, \delta_2')$ and combining the results of A, B, C, we obtain: for every $\varepsilon < \delta_0$,

1)
$$z'(\varepsilon,\tau) - z_0'(\tau) \in U \quad \text{for } \tau \in [0, \theta_0];$$

2)
$$z'(\varepsilon,\tau) - z_0'(\tau) \in V \quad \text{for } \tau \in [\theta_0, 2\theta_0].$$

3. We prove the theorem. Let a neighborhood of zero

$$U \subset L_1 \times \cdots \times L_k$$

and $t_1 > t_0$ be given. Find V, δ_0, θ_0 from Lemma 2; for V find θ_1 and δ_1 from Lemma 1. Denote-

than

$$\delta = \min\left(\delta_0, \delta_1, \left(\frac{t_1 - t_0}{2\theta_1}\right)^{1/n_1}\right).$$

Applying now Lemma 2 successively for $t^* = t_0^* = t_0 + \delta^{n_1} 2\theta$, $t_0^* + \delta^{n_1} 2\theta_0, \dots, t_0^* + n\delta^{n_1} 2\theta_0$ (here n is such that $t_0^* + n\delta^{n_1} 2\theta_0 > T$, i.e., for example,

$$n = 1 + E\left(\frac{T - t_0}{\delta^{n_1} 2\theta_2}\right),$$

we obtain: for every $\varepsilon < \delta$ and every $t \in [t_1, T]$

$$z(\varepsilon, t) - z_0(t) \in U.$$

IV. 1. As $\varepsilon \to -0$, the theorem is true if, in the condition of the theorem, asymptotic stability in the positive direction is replaced by the same stability in the negative direction.

2. The theorems of I. S. Gradshtein ($^{2-5}$) are special cases of the result proved (for finite-dimensional L_i; among them, (^{2}) for $k = 1,\ n_1 = 1$; (^{3}) for $k = 2,\ n_1 = 1,\ n_2 = 0$).
3. For locally convex linear topological spaces, the theorems on existence, uniqueness, asymptotic stability, and continuous dependence on the right-hand sides are known (^{6}).

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REFERENCES

[1] A. N. Tikhonov, *Matem. sborn.*, **27** (69), No. 1 (1950).
[2] I. S. Gradshtein, DAN, **65**, No. 6 (1949).
[3] I. S. Gradshtein, DAN, **66**, No. 5 (1949).
[4] I. S. Gradshtein, DAN, **81**, No. 6 (1951).
[5] I. S. Gradshtein, DAN, **82**, No. 1 (1952).
[6] V. M. Millionshchikov, DAN, **131**, No. 3 (1960).

Note: Figure translations are in progress. See original paper for figures.

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