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ON STATIONARY GAUSSIAN SEQUENCES POSSESSING THE PROPERTY OF STRONG MIXING

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Abstract

Full Text

MATHEMATICS

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ON STATIONARY GAUSSIAN SEQUENCES POSSESSING THE PROPERTY OF STRONG MIXING

(Presented by Academician V. I. Smirnov, 27 VI 1962)

Let $\{x_n\}$, $n = \ldots, -1, 0, 1, \ldots$, be a stationary Gaussian sequence. Denote by $\mathfrak{M}_a^b$ the minimal σ-algebra of events generated by the random variables $(x_a, x_{a+1}, \ldots, x_b)$. Put

$$\alpha(n) = \sup_{A \in \mathfrak{M}_{-\infty}^0,\, B \in \mathfrak{M}_n^\infty} |P(AB) - P(A)P(B)|. \tag{1}$$

The sequence $\{x_n\}$ is said to possess the property of strong mixing if, for it, $\alpha(n) \to 0$ as $n \to \infty$ ([1]).

Below we give several theorems on the properties of the spectral density $f(\lambda)$ of a sequence $\{x_n\}$ possessing the property of strong mixing.

1. Theorem 1. *The spectral density $f(\lambda)$ of a sequence $\{x_n\}$ possessing the property of strong mixing is representable in the form*

$$f(\lambda) = |P(\lambda)|^2 g(\lambda), \tag{2}$$

where $P(\lambda)$ is a trigonometric polynomial, and the antiderivative $G(\lambda)$ of the function $g(\lambda)$ satisfies the following condition: as $h \to 0$, uniformly in λ,

$$G(\lambda + h) + G(\lambda - h) - 2G(\lambda) = o(G(\lambda + h) - G(\lambda)). \tag{3}$$

In the note ([2]) it is proved that

$$\alpha(n) \le \rho(n) \le 2\pi\alpha(n),$$

where

$$\rho(n) = \sup_{\varphi, \psi} \left| \int_{-\pi}^{\pi} \varphi(e^{i\lambda}) \psi(e^{i\lambda}) e^{in\lambda} f(\lambda)\, d\lambda \right|, \tag{4}$$

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and the supremum is taken over all continuous functions $\varphi(e^{i\lambda})$, $\psi(e^{i\lambda})$, analytically continuable inside the unit circle, for which

$$\int_{-\pi}^{\pi} |\varphi(e^{i\lambda})|^2 f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} |\psi(e^{i\lambda})|^2 f(\lambda)\, d\lambda = 1.$$

Equality (4) makes it possible to give a purely analytic characterization of the class of spectral densities $f(\lambda)$ under investigation; it is used in an essential way in the proof of Theorem 1.

In a neighborhood of those points λ where $0 < m \leq g(\lambda) \leq M < \infty$, condition (3) simply means that $G(\lambda)$ is a smooth function ([3]). Hence, taking into account that the derivative of a smooth function has no discontinuities of the first kind, we obtain the following result ([4]):

Corollary 1. *If $\alpha(n) \to 0$ as $n \to \infty$, then the spectral density $f(\lambda)$ has no discontinuities of the first kind.*

As for those points λ at which $g(\lambda)$ tends to zero or is unbounded, for them condition (3) means that the tending to zero or to infinity occurs very slowly: whatever the point $\lambda_0 \in [-\pi, \pi]$,

$$\lim_{\lambda \to \lambda_0} \left| \frac{\ln g(\lambda)}{\ln |\lambda - \lambda_0|} \right| = 0. \tag{5}$$

Indeed, it follows from (3) that for all λ the function $h\,\Psi_\lambda(h) = \dfrac{G(\lambda+h) - G(\lambda)}{h}$ is slowly varying in the sense of Karamata. From this (5) is already easily derived. From equality (5) the following further results of the note immediately follow[4]:

Corollary 2. If $\alpha(n) \to 0$ as $n \to \infty$, then for all $\delta > 0$

$$\lim_{\lambda \to \lambda_0} |\lambda - \lambda_0|^\delta f(\lambda) = 0.$$

Corollary 3. If $\alpha(n) \to 0$ as $n \to \infty$, then the lower order of the zero λ_0 (see (4)) of the spectral density $f(\lambda)$ is necessarily a nonnegative even integer.

It is possible that all Gaussian stationary sequences whose spectral densities satisfy conditions (2), (3) possess the strong mixing property. The author is at present able to prove only the following:

Theorem 2. Let the spectral density $f(\lambda)$ of the sequence $\{x_n\}$ be representable in the form (2); suppose further that

$$\sum_{k=0}^{\infty} \omega_G^2(2^{-k}) < \infty,$$

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where

$$\omega_G(h) = \sup_{t \le h} \sup_{\lambda} \frac{|G(\lambda+t) + G(\lambda-t) - 2G(\lambda)|}{|G(\lambda+t) - G(\lambda)|}.$$

Then the sequence $\{x_n\}$ possesses the strong mixing property, and

$$\alpha(n) = O\left(\left(\sum_{k=0}^{\infty} \omega_G^2(2^{-k} n^{-1})\right)^{1/2}\right).$$

2. In this section we shall give theorems that make it possible to describe completely the class of those spectral densities $f(\lambda)$ for which $\alpha(n)$ decreases sufficiently rapidly, no more slowly than $n^{-\gamma}$, $\gamma > 0$.

Theorem 3. In order that a stationary Gaussian sequence $\{x_n\}$ possess the strong mixing property with $\alpha(n) = O(n^{-r-\beta})$, $r \ge 0$ an integer, $0 < \beta < 1$, it is necessary and sufficient that its spectral density $f(\lambda)$ be representable in the form $f(\lambda) = |P(\lambda)|^2 g(\lambda)$, where $P(\lambda)$ is a trigonometric polynomial, and the function $g(\lambda)$ is positive, $g(\lambda) \ge m > 0$, and has an r-th derivative satisfying a Hölder condition of order β.

The proof of this and of the following theorems rests on Theorem 1, certain well-known facts from the theory of approximation of functions, and the two following lemmas, the first of which is almost obvious.

Lemma 1. If the spectral density $f(\lambda)$ of the sequence $\{x_n\}$ is positive, then

$$\alpha(n) = O(E_{n-1}(f)),$$

where $E_n(f)$ denotes the best approximation of the function $f(\lambda)$ by trigonometric polynomials of degree $\le n$.

Lemma 2. If $f(\lambda)$ is the spectral density of the sequence $\{x_n\}$, then

$$E_n(f) = O\left(\sum_{k=0}^{\infty} \alpha(2^k n)\right).$$

In particular, if $\sum_{k=0}^{\infty} \alpha(2^k) < \infty$, then the spectral density $f(\lambda)$ is continuous.

Theorem 4. In order that $\alpha(n) = O(e^{-cn})$, $c > 0$, it is necessary and sufficient that the spectral density $f(\lambda)$ admit an analytic continuation to the strip of values of the complex argument $z = \lambda + i\mu$ of width $2c$.

Theorem 5. In order that $\alpha(n) = O(e^{-cn})$ for all $c > 0$, it is necessary and sufficient that the analytic continuation of $f(\lambda)$ be an entire function of z.

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Remark. Many of the results listed here extend to processes $\{x_t, -\infty < t < \infty\}$ with continuous time. In this case the polynomials $P(\lambda)$ should be replaced by entire functions of finite degree.

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Note: Figure translations are in progress. See original paper for figures.

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