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1962

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Abstract

Full Text

MATHEMATICS

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ON UNIONS OF STRONGLY PARACOMPACT SPACES

(Presented by Academician P. S. Aleksandrov on 30 III 1962)

1. It is known that if a topological space has a locally finite open covering consisting of sets with paracompact closures, then it is itself paracompact. An analogous assertion is true also for many other properties. The following proposition holds (in which a class of spaces is called F-hereditary if it contains, together with every space, all its closed subspaces):

Proposition 1. Let V be an F-hereditary class of normal spaces and suppose: a) if P is a union of disjoint open subspaces from the class V, then $P \in V$; b) if $P = G_1 \cup G_2$, the G_i are open, $\overline{G_i} \in V$, then $P \in V$. Let the space S have a locally finite open covering consisting of spaces with closures belonging to V. Then S belongs to the class V.

Proof. Let there be a covering G with the indicated properties. Insert into it a locally finite open covering of the form $\bigcup_{n=1}^{\infty} A_n$, where the A_n are discrete systems. Denote by H_n the union of the sets $A \in A_n$; obviously, $\overline{H_n} \in V$. Insert into the countable locally finite covering $\{H_n\}$ a star-finite open covering (see (2)). Each component U of this covering* has the form

$$U = \bigcup_{k=1}^{\infty} U_k,$$

where the U_k are open, $\overline{U_k} \in V$, $\overline{U_k} \subset U_{k+1}$. Put

$$G_k = \overline{U_{k+1}} - U_k, \qquad M = \bigcup_{k=1}^{\infty} G_{2k}, \qquad N = \bigcup_{k=1}^{\infty} G_{2k+1}.$$

Then M, N, and $U = M \cup N$ belong to the class V, whence it follows that $S \in V$.

Remark. If the class V satisfies conditions a), b), then the class of all spaces hereditarily belonging to V also satisfies them.

2. The class of strongly paracompact spaces** satisfies condition a) of Proposition 1, but not condition b); as an example one may take the product of an open interval and the "point hedgehog" ***.

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We shall now construct, for $n = 2, 3, \dots$, a metric space P such that:

1) P is a union of n, but not of $n-1$, open subsets with strongly paracompact closures;

2) P is a union of two strongly paracompact closed subsets.

Let S be an $(n-1)$-dimensional cube lying in the plane $x_n = 0$ of the space E_n. By known theorems there exists $\varepsilon > 0$ such that if M_i

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* By a component U of a covering $\mathcal{U}$ we mean the union of the sets of a maximal subsystem $\mathcal{U}' \subset \mathcal{U}$ such that for any $G, H \in \mathcal{U}'$ there exist $G_1, \dots, G_k \in \mathcal{U}$ with $G \cap G_1 \neq \emptyset$, $G_i \cap G_{i+1} \neq \emptyset$ $(i = 1, \dots, k-1)$, $G_k \cap H \neq \emptyset$.

** A Hausdorff space is called strongly paracompact if into every open covering of it one can insert a star-finite open covering.

*** By a "point hedgehog" we mean a metrizable space all of whose points are isolated except one, every neighborhood of which is uncountable.

open in S, $S = \bigcup_{i=1}^{n-1} M_i$, then at least one of the components of the sets M_i has diameter $> \varepsilon$. Denote by $\mathcal{Y}$ the interval $\langle -1, 1 \rangle$, and by Φ the set of all its homeomorphic images f in S such that the diameter of $f(\mathcal{Y})$ is not less than ε. Take some abstract set A and its mapping φ onto Φ such that all $\varphi^{-1}(f)$ are uncountable; instead of $\varphi(a)$ we shall write φ_a. For $a \in A$ denote by T_a the set of all points $[x_1, \dots, x_n, a] \in E_n \times A$ such that $[x_1, \dots, x_{n-1}, 0] = \varphi_a\left(\sin \frac{1}{x_n}\right)$; put $C_a = \varphi_a(\mathcal{Y}) \cup T_a$, and denote by σ_a the metric on C_a obtained by means of the obvious embedding in E_n. We now introduce a metric ρ on $P = \bigcup_{a \in A} C_a$, setting $\rho(x, y) = \sigma_a(x, y)$, if x, y belong to one and the same C_a; $\rho(x, y) = \inf(\sigma_a(x, z) + \sigma_\beta(z, y))$, if $x \in C_a,\ y \in C_\beta,\ a \neq \beta$.

Denote by M and, respectively, by Q the subset of the space P consisting of all $x \in S$ and all points of the form $[x_1, \dots, x_n, a] \in T_a$ satisfying the condition $\sin \frac{1}{x_n} \geq -\frac{1}{2}$ (respectively, $\sin \frac{1}{x_n} \leq \frac{1}{2}$). Obviously, $P = M \cup Q$, and M, Q are closed; it is easy to establish that M, Q are strongly paracompact.

The space P is not the union of $n-1$ open subsets: if $P = \bigcup_{i=1}^{n-1} G_i$ with G_i open, then at least one of the sets $G_i \cap S$ has a component of diameter $> \varepsilon$, and therefore contains some $f(\mathcal{Y})$, $f \in \Phi$; then, for a suitable $\delta > 0$, the set consisting of all points $f(\mathcal{Y})$ and all $[x_1, \dots, x_n, a] \in T$ such that $\varphi_a = f,\ 0 \leq x_n \leq \delta$, is contained in G_i, is closed in P, but is not strongly paracompact.

Finally, let us show that P is the union of n open subsets with strongly paracompact closures. From known theorems it follows that $S = \bigcup_{i=1}^{n} H_i$, where H_i are open in S and their components have diameter $< \varepsilon$. Denote by Q_i the set consisting of all $x \in H_i$ and all $[x_1, \dots, x_n, a]$ such that $[x_1, \dots, x_{n-1}, 0] \in H_i$. Then the H_i are open and their closures are strongly paracompact.

3. It is easy to verify the following propositions:

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Proposition 2. Let the space P be the union of a locally finite system of order $\leq n$ of open subsets with strongly paracompact closures. Then P is the union of $n+1$ such subsets.

Proposition 3. Let the space P be the union of a locally finite system of open subsets with strongly paracompact closures. Then P is the union of a locally finite countable system of such sets.

Proposition 4. A space P is paracompact and locally strongly paracompact if and only if it is the union of a countable locally finite system of its open subsets with strongly paracompact closures.

4. As is known (see also the example given above), a space that is the union of two closed strongly paracompact subspaces need not be strongly paracompact. It may even fail to be locally strongly paracompact, as is shown by example 3 from the article ([3]) (to which Yu. M. Smirnov drew the author' s attention). However, the following assertion holds:

Proposition 5. If $P = M_1 \cup M_2$, M_1, M_2 are closed and strongly paracompact, and the space $M_1 \cap M_2$ is locally Lindelöf*, then P is strongly paracompact.

* A space is called Lindelöf if from each of its open coverings one can choose a countable one.

Proof. Put $M_1 = A_1$, $\overline{P - M_1} = A_2$, $Q = A_1 \cap A_2$. Then Q is a locally Lindelöf paracompact space, and therefore there exists a discrete system $G = \{G_\lambda; \lambda \in \Lambda\}$ of open-and-closed subsets of Q such that $\bigcup_{\lambda\in\Lambda} G_\lambda = Q$ and each G_λ is a Lindelöf space ([5]). Let $\mathcal{W}$ be an open covering of P; it is enough to inscribe in it a star-countable open covering ([3]). P is paracompact, and therefore there exists an open covering $\mathcal{U}$ of it such that no $U \in \mathcal{U}$ meets two sets G_λ and the system $\{\operatorname{st} U;\ U \in \mathcal{U}\}^*$ is inscribed in $\mathcal{W}$. Put $\mathcal{U}_i = \{U \cap A_i;\ U \in \mathcal{W}\}$, $i = 1, 2$. Let $\mathcal{V}_i$ be a star-countable open (in A_i) covering of the space A_i, inscribed in $\mathcal{U}_i$. Let $\mathcal{V}_{\lambda,i}$ be a countable subsystem of $\mathcal{V}_i$ which covers G_λ. If now we put

$$\mathcal{V}_\lambda = \{\operatorname{Int}(V_1 \cup V_2);\ V_1 \in \mathcal{V}_{\lambda,1},\ V_2 \in \mathcal{V}_{\lambda,2}\},$$

then it is clear that $\mathcal{V}_\lambda$ is a countable system of open subsets of P covering G_λ. If, moreover, we put

$$\mathcal{X}_i = \{V - Q;\ V \in \mathcal{V}_i - \bigcup_{\lambda\in\Lambda} \mathcal{V}_{\lambda,i}\},\ i = 1, 2; \qquad \mathcal{V} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \bigcup_{\lambda\in\Lambda} \mathcal{V}_\lambda,$$

then, obviously, $\mathcal{V}$ is a star-countable open covering of P, inscribed in $\mathcal{W}$.

5. We now give an answer to the problem of the additivity of the class of strongly paracompact spaces with respect to open subsets (the author' s attention was also drawn to this problem by Yu. M. Smirnov), namely, we

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shall construct a paracompact but not strongly paracompact space P and its open strongly paracompact subspaces G_1, G_2, such that $P = G_1 \cup G_2$. Let T_{ω_1} be the set of all countable ordinal numbers. Let $M = T_{\omega_1} \times \langle 0, 1 \rangle$. On M we take the lexicographic order $<$ and the topology determined by this order. Also denote $C = \{[x, 0];\ x \in T_{\omega_1}\}$, $D = \{[x, \frac{1}{2}];\ x \in T_{\omega_1}\}$. Let A be an uncountable discrete space, $Q = (\xi) \cup (M \times A)$; on Q we take the following topology: on $M \times A$ we take the product topology of the spaces; a complete system of neighborhoods of the point ξ will be the system $\{U_x;\ x \in M\}$, where $U_x = \{[m, a] \in M \times A :\ x < m\} \cup (\xi)$.

Put also

$$S_1 = (\xi) \cup [(M - C) \times A], \qquad S_2 = (\xi) \cup [(M - D) \times A],$$

$$S_3 = (\xi) \cup [(M - C \cup D) \times A].$$

Put $P^* = Q \times \langle 0, 1 \rangle$ and, as the space P with the required properties, take the following subspace of it:

$$P = (S_1 \times \{0\}) \cup (S_3 \times (0, 1)) \cup (S_2 \times \{1\}).$$

It can be verified that P is not strongly paracompact, whereas the sets

$$G_1 = (S_1 \times \{0\}) \cup (S_3 \times (0, 1))$$

and

$$G_2 = (S_3 \times (0, 1)) \cup (S_2 \times \{1\})$$

are strongly paracompact.

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Received
28 III 1962

References

1. M. Katetov, *Czechoslovak Mathematical Journal*, **2** (77), 333 (1952).

2. K. Morita, *Math. Japon.*, **1**, 60 (1948).

3. Yu. M. Smirnov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **20**, 253 (1956).

4. Yu. M. Smirnov, *Fund. Math.*, **43**, 387 (1956).

5. V. Šedivá, *Czechoslovak Mathematical Journal*, **9** (84), 50 (1959).

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* $\mathrm{st}\, U$ denotes the union of all sets of the covering $\mathcal{U}$ having nonempty intersection with U.

Note: Figure translations are in progress. See original paper for figures.

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