Mathematics

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Abstract

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On the Approximation of a Plane Wave by Superpositions of Plane Waves of Prescribed Directions

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Let $\mathbf{x} = (x_1, \dots, x_n)$ be a point of the Euclidean space R_n, and let D be some domain of this space. Further, let $\mathbf{a} = (a_1, \dots, a_n)$ be a point of the $(n-1)$-dimensional real projective space Π_{n-1}, given by homogeneous coordinates $a_1, \dots, a_n$. In the space Π_{n-1} take a set of points M and an arbitrary point $\mathbf{a}$. Let $f(t)$ be an arbitrary function continuous on the interval

$$\inf_{\mathbf{x}\in D}(\mathbf{ax}) < t < \sup_{\mathbf{x}\in D}(\mathbf{ax}), \qquad \mathbf{ax} = a_1x_1 + \dots + a_{nx}n. \tag{1}$$

We shall derive necessary and sufficient conditions under which every function $f(\mathbf{ax})$, $\mathbf{x} \in D$, can be uniformly approximated inside the domain D by sums of the form

$$\sum_{i=1}^{N} \varphi_i(\mathbf{a}_i\mathbf{x}), \tag{2}$$

where N is an arbitrary natural number, $\mathbf{a}_i$ is a point of the set M, and $\varphi_i(t_i)$ is a function continuous on the interval

$$\inf_{\mathbf{x}\in D}(\mathbf{a}_i\mathbf{x}) < t_i < \sup_{\mathbf{x}\in D}(\mathbf{a}_i\mathbf{x}), \qquad \mathbf{a}_i\mathbf{x} = a_{i1}x_1 + \dots + a_{in}x_n.$$

For this purpose let us consider an arbitrary homogeneous polynomial $P(\mathbf{y}) = P(y_1, \dots, y_n)$ in the variables $y_1, \dots, y_n$ with real coefficients and introduce the following definition.

Definition. We shall say that a point $\mathbf{a}$, $\mathbf{a} \in \Pi_{n-1}$, is **algebraically dependent** on the set M if every polynomial $P(\mathbf{y})$ containing the set M (i.e., vanishing at each of the points of the set M) also contains the point $\mathbf{a}$.

Theorem. In order that an arbitrary continuous function $f(\mathbf{ax})$, $\mathbf{x} \in D$, be uniformly approximable inside the domain D by sums of the form (2), it is necessary and sufficient that the point $\mathbf{a} = (a_1, \dots, a_n)$ be algebraically dependent on the set M.

Proof of necessity. Suppose that an arbitrary continuous function $f(\mathbf{ax})$ can be uniformly approximated inside the domain D by sums of the form (2). Let $P(y_1, \dots, y_n)$ be an arbitrary polynomial of degree m containing the set M. Take some closed ball $\overline{K}$, lying together with its boundary in the domain D. By $v_{\overline{K}}$ denote the class of all functions $v(\mathbf{x})$, m times continuously differentiable in the ball $\overline{K}$, equal to zero together with their partial derivatives up to order $m-1$ inclusive on the boundary of the ball $\overline{K}$. For functions of the class $v_{\overline{K}}$ define the functional

$$(u, v) = \int_{\overline{K}} u(\mathbf{x})\, P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) v(\mathbf{x})\, dx,$$

where $u(\mathbf{x})$ is an arbitrary function continuous in the ball $\overline{K}$. The functional (u, v) is additive in u and v, and, for a fixed function $v(\mathbf{x})$, is continuous with respect to uniform convergence in the class of all functions $u(\mathbf{x})$ continuous in the ball $\overline{K}$. It is also easy to see that, if $P(a_{01}, \dots, a_{0n}) = 0$, then $(\varphi(\mathbf{a}_0\mathbf{x}), v) = 0$ for functions of the class $v_{\overline{K}}$, whatever the continuous function $\varphi(\mathbf{a}_0\mathbf{x})$, $\mathbf{x} \in \overline{K}$, may be. Hence, and from the assumption made concerning the function $f(\mathbf{ax})$, it follows that $(f(\mathbf{ax}), v) = 0$, if $v(\mathbf{x}) \in v_{\overline{K}}$. Since the function $f(t)$ is arbitrary, choose it so that on the interval

$$\inf_{\mathbf{x} \in \overline{K}} (\mathbf{ax}) \leq t \leq \sup_{\mathbf{x} \in \overline{K}} (\mathbf{ax})$$

it has a continuous derivative $f^{(m)}(t)$, and moreover $f^{(m)}(t) \neq 0$ on this interval. Integrating by parts, we obtain

$$(f, v) = (-1)^m \int_D f^{(m)}(\mathbf{ax})\, P(a_1, \dots, a_n)\, v(\mathbf{x})\, d\mathbf{x} = 0,$$

whence, by virtue of the arbitrariness of the function $v(\mathbf{x}) \in v_{\overline{K}}$, it follows that

$$f^{(m)}(\mathbf{ax})\, P(a_1, \dots, a_n) \equiv 0, \qquad \mathbf{x} \in \overline{K},$$

i.e. $P(a_1, \dots, a_n) = 0$, and the point $\mathbf{a} = (a_1, \dots, a_n)$ is algebraically dependent on the set M.

Proof of sufficiency. Since every function continuous on the interval (1) can be uniformly approximated within this interval by polynomials, in order to prove our assertion it is enough, for every natural m, to prove the possibility of the representation

$$(\mathbf{ax})^m = \sum_{i=1}^{k} \lambda_i (\mathbf{a}_i \mathbf{x})^m, \qquad \mathbf{a}_i \in M, \tag{3}$$

where k is some natural number; $\lambda_1, \dots, \lambda_k$ are real numbers.

Obviously, the numbers λ_i must satisfy the system of equations

$$a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} = \sum_{i=1}^{k} a_{i1}^{m_1} a_{i2}^{m_2} \cdots a_{in}^{m_n} \lambda_i, \tag{4}$$

$$m_j \geq 0, \qquad m_1 + m_2 + \dots + m_n = m.$$

Take $k = C^{n-1}_{m+n-1}$ and consider the determinant of the resulting system (4)

$$\Delta(\mathbf{a}_1, \dots, \mathbf{a}_k) = \left| a_{i1}^{m_1} a_{i2}^{m_2} \cdots a_{in}^{m_n} \right|$$

for all possible collections of points belonging to the set M. If, for the given m, apart from the identically zero one, there are no polynomials of degree m containing the set M, then, as is easy to see, in the set M there will be found a collection of points $\mathbf{a}_i^*$, $i = 1, \dots, k$, for which $\Delta(\mathbf{a}_1^*, \dots, \mathbf{a}_k^*) \neq 0$, and the possibility of the representation (3) for this m is proved. If such a nontrivial polynomial does exist, then $\Delta(\mathbf{a}_1, \dots, \mathbf{a}_k) = 0$, whatever the collection of points $\mathbf{a}_i$, $i = 1, \dots, k$, from the set M may be. In this case, among the minors of the determinant $\Delta(\mathbf{a}_1, \dots, \mathbf{a}_k)$ there will be at least one minor $\Delta(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_s})$, $s < k$, and in the set M at least one system of points $\mathbf{a}_{i_1}^*, \dots, \mathbf{a}_{i_s}^*$ such that $\Delta(\mathbf{a}_{i_1}^*, \dots, \mathbf{a}_{i_s}^*) \neq 0$, while all minors of higher order of the determinant $\Delta(\mathbf{a}_1, \dots, \mathbf{a}_k)$ will be equal to zero for all possible collections of points from M.

If, among the rows of the minor $\Delta(\mathbf{a}_{i_1}^*, \dots, \mathbf{a}_{i_s}^*)$, there is no row of the form

$$a_{i1}^{m_1^0} a_{i2}^{m_2^0} \cdots a_{in}^{m_n^0}, \qquad m_1^0 + m_2^0 + \dots + m_n^0 = m, \qquad m_j^0 \geq 0,$$

then we border the minor

$\Delta(a_{i_1}^*, \dots, a_{i_s}^*)$ by means of the elements of this row and the elements of any column of the original determinant, taken for an arbitrary point $a_i \in M$. We obtain a certain minor $\Delta(a_{i_1}^*, \dots, a_{i_s}^*, a_i)$. Fix the points $a_{i_1}^*, \dots, a_{i_s}^*$, and let the point a_i run through all points of the set M. By construction, each time

$$\Delta(a_{i_1}^*, \dots, a_{i_s}^*, a_i) = 0.$$

Expanding this minor with respect to the elements of the appended column, we find that on the set M, between the monomial

$$a_{i1}^{m_1^0} a_{i2}^{m_2^0} \dots a_{in}^{m_n^0}$$

and the monomials corresponding to the rows of the minor $\Delta(a_{i_1}^*, \dots, a_{i_s}^*)$, there is a linear dependence: the monomial

$$a_{i1}^{m_1^0} a_{i2}^{m_2^0} \dots a_{in}^{m_n^0}$$

is expressed linearly in terms of the remaining indicated monomials. To the linear combination obtained we associate the homogeneous polynomial $P(y_1, \dots, y_n)$, replacing in the determinant $\Delta(a_{i_1}^*, \dots, a_{i_s}^*, a_i)$ the components $a_{i1}, a_{i2}, \dots, a_{in}$ of the point a_i respectively by the variables $y_1, y_2, \dots, y_n$. Since the polynomial $P(y)$ contains the set M, by the hypothesis of the theorem it must also contain the point a. Hence it follows that the function $u = (ax)^m$ satisfies the partial differential equation

$$P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) u = 0.$$

Substituting the monomial $(ax)^m$ into this equation, we find that between the monomial

$$a_1^{m_1^0} a_2^{m_2^0} \dots a_n^{m_n^0}$$

and the monomials corresponding to the rows of the minor $\Delta(a_{i_1}^*, \dots, a_{i_s}^*)$, taken at the point a, there exists exactly the same linear dependence as that found above for the points $a_i \in M$. Since the row chosen of the form

$$a_{i1}^{m_1^0} a_{i2}^{m_2^0} \dots a_{in}^{m_n^0},$$

which was not included among the rows of the determinant $\Delta(a_{i_1}^*, \dots, a_{i_s}^*)$, was arbitrary, the proved result implies the solvability of the system (4) with respect to λ_i and the possibility of the representation (3).

The theorem just given makes it possible to formulate the concept of algebraic dependence of a point on a set M also as follows:

A point $a \in \Pi_{n-1}$ is algebraically dependent on the set M if there exists a domain D such that an arbitrary function $f(ax)$, $x \in D$ ($f(t)$ continuous on the interval (1)), can be uniformly approximated inside D by sums of the form (2).

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Note: Figure translations are in progress. See original paper for figures.

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