



Soviet-era science, translated into English

MATHEMATICS

1962

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Abstract

Full Text

MATHEMATICS

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ON THE SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS WITH STRONG NONLINEARITIES

(Presented by Academician V. I. Smirnov on 19.V.1962)

The paper studies the first and second boundary-value problems for quasilinear elliptic equations of the following form:

$$\mathcal{L}u \equiv - \sum_{i=1}^m \frac{\partial}{\partial x_i} (\varphi(T)p_i) = f(x), \quad p_i = \frac{\partial u}{\partial x_i}, \quad T = |\text{grad } u|. \quad (1)$$

The results known so far concern the case of power growth of the coefficients with respect to the first derivatives (¹⁻⁴). In this setting, the natural assumption is that the order of growth decreases by one under differentiation. In the case considered by us, strong growth of the function $\varphi(T)$ is admissible.

We shall assume that $\varphi(T)$ satisfies the following conditions:

A. $\varphi'(T) > 0, \quad \varphi(0) = p_0 > 0.$

B. $\varphi'(T)T \leq M\varphi^{1+k}(T), \quad k \leq \min(1, \frac{4}{m}).$

For equation (1) we pose the first boundary-value problem

$$u|_{\Gamma} = 0 \quad (2)$$

and the second boundary-value problem

$$\frac{\partial u}{\partial n} \Big|_{\Gamma} = 0. \quad (3)$$

Here $f(x_i) \in L_2(\Omega)$; Ω is a bounded domain of m -dimensional Euclidean space with a twice continuously differentiable boundary Γ . In considering the first boundary-value problem, the domain Ω is assumed to be convex.

Theorem. Under conditions A and B there exists a unique generalized solution of each of the boundary-value problems (1)–(2), (1)–(3), and for it the following estimates hold:

$$\int_{\Omega} \varphi(T) \left(T^2 + \sum_{i,j} u_{ij}^2 \right) dx \leq C_1 \|f\|_{\mathcal{L}^2}^2; \quad (4)$$

$$\int_{\Omega} \varphi^2(T) \sum_{i,j} u_{ij}^2 dx \leq C_2 \|f\|_{\mathcal{L}^2}^2; \quad (5)$$

$$\int_{\Omega} \varphi^2(T) dx \leq C_3 + C_4 \|f\|_{\mathcal{L}^2}^{\frac{4}{2-k}}. \quad (6)$$

By a generalized solution of the first boundary-value problem we mean a function from \dot{W}_2^2 satisfying the equation almost everywhere.

Under the boundary condition (3), a generalized solution is defined as a function from W_2^2 satisfying the integral identity and the equation almost everywhere. It is assumed that $\int_{\Omega} f dx = 0$, and the solution is sought in that class.

For the proof of the theorem the following method is used. In equation (1) we introduce a numerical parameter α :

$$\mathcal{L}_{\alpha} u \equiv - \sum_{i=1}^m \frac{\partial}{\partial x_i} \left(\frac{\varphi}{1 + \alpha\varphi} p_i \right) = f(x), \quad 0 < \alpha < \alpha_0. \quad (7)$$

In this case the equation remains elliptic, and the ellipticity constant does not depend on α , while the coefficients of equation (7) are bounded for each fixed α (by virtue of condition B).

For equations with bounded nonlinearities the solvability of boundary-value problems is known (see, for example, (5)).

To carry out the passage to the limit with respect to α as $\alpha \rightarrow 0$, estimates are proved for generalized solutions of the auxiliary boundary-value problems (7)–(2), (7)–(3). This method was first applied by the author (4) to the study of quasilinear equations with power nonlinearities.

Uniform estimates with respect to α are established in Lemmas 1 and 2.

Lemma 1. The generalized solution of each of the boundary-value problems (7)–(2), (7)–(3) satisfies the inequality:

$$\int_{\Omega} \frac{\varphi}{1 + \alpha\varphi} \left(T^2 + \sum_{i,j} u_{ij}^2 \right) dx + \int_{\Omega} \frac{\varphi'}{(1 + \alpha\varphi)^2} \frac{\sum_i (\sum_j p_j u_{ij})^2}{T} dx \leq C \|f\|_{\mathcal{L}^2}^2. \quad (8)$$

For the proof of the lemma we integrate twice by parts the expression $\mathcal{L}_\alpha u \cdot (-\Delta u)$ (we first assume the function u to be smooth):

$$\begin{aligned} & \sum_{i,j} \int_{\Omega} \frac{\partial}{\partial x_i} \left(\frac{\varphi}{1 + \alpha\varphi} p_i \right) \frac{\partial}{\partial x_j} (p_j) dx = \\ &= \int_{\Omega} \frac{\varphi}{1 + \alpha\varphi} \sum_{i,j} u_{ij}^2 dx + \int_{\Omega} \frac{\varphi'}{(1 + \alpha\varphi)^2} \frac{\sum_j (\sum_i p_i u_{ij})^2}{T} dx + \\ &+ \int_{\Gamma} \frac{\varphi}{1 + \alpha\varphi} \left(\Delta u \frac{\partial u}{\partial n} - \frac{\partial}{\partial n} (T^2) \right) d\Gamma. \end{aligned} \quad (9)$$

To estimate the boundary integral we use a device of O. A. Ladyzhenskaya (6). Under the boundary condition of the second kind, the expression

$$\Delta u \frac{\partial u}{\partial n} - \frac{\partial}{\partial n} (T^2) \quad (10)$$

is equal to zero on the boundary.

Under the first boundary condition, the difference (10), in local curvilinear coordinates (y_1, y_2, \dots, y_n) (y_n is directed along the exterior normal to Γ), is transformed into the form $-\left(\frac{\partial u}{\partial n}\right)^2 \Delta\omega$. Here $y_n = \omega(y_1, y_2, \dots, y_{n-1})$ is the equation of the portion of the boundary near the point under consideration M .

In the case of a convex domain $\Delta\omega$ is negative and, consequently, in estimates from below of the right-hand side of (9) the boundary integral may be discarded.

Thus,

$$\int_{\Omega} \frac{\varphi}{1 + \alpha\varphi} \sum_{i,j} u_{ij}^2 dx + \int_{\Omega} \frac{\varphi'}{(1 + \alpha\varphi)^2} \frac{\sum_j (\sum_i p_i u_{ij})^2}{T} dx \leq C \|\mathcal{L}_\alpha u\|_{\mathcal{L}^2}^2. \quad (11)$$

Taking into account the boundedness of $\frac{\varphi}{1 + \alpha\varphi}$ and $\frac{\varphi' T}{(1 + \alpha\varphi)^2}$, one can show that estimate (11), proved for smooth u , also holds for a generalized solution.

The estimate of the integral

$$\int_{\Omega} \frac{\varphi}{1 + \alpha\varphi} T^2 dx$$

is obtained from the integral identity for the generalized solution.

Lemma 2. *The generalized solution of each of the boundary-value problems (7)–(2), (7)–(3) satisfies the inequalities:*

$$\int_{\Omega} \frac{\varphi^2}{(1 + \alpha\varphi)^2} \sum_{i,j} u_{ij}^2 dx + \int_{\Omega} \frac{\varphi\varphi'}{(1 + \alpha\varphi)^3} \frac{\sum_j (\sum_i p_i u_{ij})^2}{T} dx + \int_{\Omega} \frac{(\varphi')^2}{(1 + \alpha\varphi)^4} \frac{(\sum_{i,j} p_i p_j u_{ij})^2}{T^2} dx \leq C_1 \|f\|_{L_2}^2; \quad (12)$$

$$\int_{\Omega} \frac{\varphi^2}{(1 + \alpha\varphi)^2} dx \leq C_2 + C_3 \|f\|_{L_2}^{\frac{4}{2-k}}. \quad (13)$$

The first estimate is obtained analogously to (8), by integration by parts of the expression $\mathcal{L}_\alpha u \cdot \mathcal{L}_\alpha u$.

Let us dwell in more detail on the proof of estimate (13). From inequality (12) it follows that the function

$$\Phi(T) \equiv \int_0^T \sqrt{\frac{\varphi\varphi'T}{(1 + \alpha\varphi)^3}} dT$$

has square-summable derivatives with respect to x_i . It is easy to show that, for all T ,

$$\Phi(T) \leq \frac{1}{2} \frac{\varphi T}{1 + \alpha\varphi},$$

and, by virtue of (8), the function $\Phi(T)$ belongs to \mathcal{L}_1 in Ω . Hence $\Phi(T) \in W_2^1$, and, according to the embedding theorem W_2^1 in \mathcal{L}_q ($q \leq \frac{2m}{m-2}$):

$$\|\Phi\|_{L_q} \leq C_4 \|\Phi\|_{W_2^1} \leq C_5 \|f\|_{L_2}.$$

From condition B it follows that

$$\left(\frac{\varphi}{1 + \alpha\varphi} \right)^2 \leq C_6 (\Phi T)^{\frac{4}{2-k}} + C_7.$$

Consequently, estimate (13) holds.

We proceed to the proof of the theorem. Let $\{\alpha_n\}$ be an arbitrary sequence converging to zero ($\alpha_n > 0$). From inequality (8) and the boundary conditions it follows that the sequence of solutions $\{u_n\}$ is bounded in W_2^2 .

Using estimates (8) and (12), from the sequence $\{u_n\}$ we choose a subsequence such that:

- 1) $u_{ij}^{(n)}$ converges weakly in \mathcal{L}_2 to u_{ij} ;
- 2) $p_i^{(n)}$ converges almost everywhere to p_i ;
- 3)

$$\sqrt{\frac{\varphi_n}{1 + \alpha_n \varphi_n}} u_{ij}^{(n)}$$

converges weakly in \mathcal{L}_2 ;

4)

$$\frac{\varphi_n}{1 + \alpha_n \varphi_n} u_{ij}^{(n)}$$

converges weakly in \mathcal{L}_2 ;

5)

$$\sqrt{\frac{\varphi_n}{T_n} \frac{\sum_i p_i^{(n)} u_{ij}^{(n)}}{1 + \alpha_n \varphi_n}}$$

converges weakly in \mathcal{L}_2 .

It is proved that the limiting functions of the sequences in 3), 4), and 5) are, respectively,

$$\sqrt{\varphi} u_{ij}, \quad \varphi u_{ij}, \quad \sqrt{\frac{\varphi'}{T} \sum_i p_i u_{ij}}.$$

Consequently, for the limiting function u the estimates (4), (5), (6) hold. Conditions 2)–5) and inequalities (12), (13) make it possible to prove that u is a generalized solution of the posed problem. The uniqueness of the solution follows from the ellipticity of equation (1).

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Received
16 V 1962

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