MATHEMATICS

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Abstract

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ON ONE SOLUTION OF THE CLASSICAL PROBLEM OF OSCILLATIONS

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1. In 1836 Sturm ([1]) posed the problem:
   For which p and q are the solutions $x(t) \not\equiv 0 \subset C^{(2)}$ of the equation

$$x'' + px' + qx = 0; \qquad p, q \subset C^{(0)} \quad \text{for } t \geq t_0 \tag{1}$$

oscillatory, i.e., does each of them have an infinite sequence of zero values?

Modern solutions of this problem are based on various applications of the following theorems, which occur in the literature in different formulations:

For oscillation of the solutions of (1), it is necessary and sufficient that there exist θ ($\theta - p/2 \subset C^{(1)}$), for which

$$\int_{t_0}^{\infty} \exp\left[-2\int_{t_0}^{\eta} \theta\, d\xi\right] d\eta = \infty; \qquad \int_{t_0}^{\infty} \exp\left[2\int_{t_0}^{\eta} \theta\, d\xi\right] Q\, d\eta = +\infty; \qquad Q \geq 0, \tag{2}$$

where

$$Q = (\theta - p/2)' + \theta^2 + q - p^2/4 \ ([2]).$$

For non-oscillation of the solutions of (1), it is necessary and sufficient that there exist θ ($\theta - p/2 \subset C^{(1)}$) and $T \geq t_0$, for which ([3])

$$Q \leq 0 \quad \text{for } t \geq T. \tag{3}$$

Determining for which of the conditions (2) or (3) the required θ exists has in a number of cases presented considerable difficulties. The improved formulation of criterion (2), given in this article, gives a direct indication of how to do this for given p and q.

2. Hartman ([4]), Wintner ([5]), Hille ([6]), Zlamal ([7]), Kondrat' ev ([8]), and many other authors obtained a large number of sufficient conditions, using criterion (3) in the manner of the well-known Picone identity ([9]) as an inexhaustible "generator" for oscillation conditions, proving each of them by contradiction. Since along this path the necessity of the condition is not verified, some such results turn out to be equivalent to (2), but with a different dependence on an arbitrary function, while others are only sufficient. Moore ([10]) proved (2) for $\theta = p/2 + f'/f$, $f > 0 \subset C^{(2)}$; Borůvka ([11]) connects conditions (2) with his theory of dispersions; Olekh, Opial, and Vazhevskii ([12]) replace them by the method of generalized polar coordinates; and, finally, Rab ([13]) publishes a survey of numerous results obtained by various authors in recent times (the bibliography contains 43 titles). This survey, taking account of corrections ([14]), gives a sufficiently complete picture of the modern set of conditions for oscillation of the solutions of (1). It remains only unnoticed that in (2) one can dispense with the condition

$$Q \geq 0$$

and that the substitution

$$\theta = \frac{p}{2} - \int_{t_0}^{t} \left(q - \frac{p^2}{4} \right) d\xi + \beta,$$

where $\beta \subset C^{(1)}$ (incidentally, the one used in ([2]), cited in ([13])), brings (2) to the theorem:

Theorem. The solutions of (1) are oscillatory if and only if

$$-\frac{p}{2} + \int_{t_0}^{t} \left(q + \frac{p^2}{4} \right) d\xi = \beta - \theta; \qquad \beta \in C^{(1)}, \quad \theta \in C^{(0)}, \tag{4}$$

where

$$\int_{t_0}^{\infty} \exp \left[-2 \int_{t_0}^{\eta} \theta \, d\xi \right] d\eta = \infty; \tag{5}$$

$$\int_{t_0}^{\infty} \exp \left[2 \int_{t_0}^{\eta} \theta \, d\xi \right] (\beta' + \theta^2) \, d\eta = +\infty. \tag{6}$$

3. For p and q of equation (1) there always exists

$$I = -\frac{p}{2} + \int_{t_0}^{t} \left(q - \frac{p}{4}\right)^2 d\xi, \tag{7}$$

which, as is known, is connected with the distribution of the zeros of its solutions. In the expansion (4),

$$\beta = \int_{t_0}^{t} \left(q - \frac{p^2}{4}\right) d\xi + \psi; \qquad \theta = \frac{p}{2} + \psi; \qquad \psi \in C^{(1)}. \tag{8}$$

Using the arbitrary choice of ψ, one must either satisfy (5) and (6), or prove that it is impossible to satisfy these conditions. In the latter case there always exist $\psi = \psi_1$, for which (8) satisfies

$$\int_{t_0}^{\infty} \left\{ \exp\left[-2\int_{t_0}^{\eta} \theta\, d\xi\right] + \exp\left[2\int_{t_0}^{\eta} \theta\, d\xi\right] (\beta' + \theta^2)_+ \right\} d\eta < \infty; \qquad f_+ = \frac{f + |f|}{2}, \tag{9}$$

and $\psi = \psi_2$, for which there will be found such a $T \geq t_0$ that

$$\beta' + \theta^2 \leq 0 \quad \text{for all } t \geq T. \tag{10}$$

3a. For every $p \in C^{(0)}$ on $t \geq t_0$ there is an $s \in C^{(1)}$ on the same interval such that

$$s - \frac{p}{2} \geq 0 \quad \text{and} \quad \int_{t_0}^{\infty} \left(s - \frac{p}{2}\right) d\xi < \infty.$$

3b. If in the expansion

$$\int_{t_0}^{t} \left(q - \frac{p^2}{4} - s'\right) d\xi = \beta - \theta,$$

in which $\beta' + \theta^2 \geq 0$, $\int_{t_0}^{t} \theta\, d\xi$ remains bounded and $\beta \to \infty$ as $t \to \infty$, or if

$$\underline{\lim} \int_{t_0}^{t} \theta\, d\xi \neq \overline{\lim} \int_{t_0}^{t} \theta\, d\xi,$$

there exists such a C that

$$\int (\beta' + \theta^2)\, dt = \infty$$

on the set where

$$\int_{t_0}^{t} \theta\, d\xi \geq C,$$

and

$$\int dt = \infty$$

on the set where

$$\int_{t_0}^{t} \theta\, d\xi \leq C,$$

then the solutions of (1) are oscillatory.

3c. If

$$\int_{t_0}^{\infty} f\, d\xi = +\infty,$$

then there exists a representation

$$\int_{t_0}^{t} f\, d\xi = \beta - \theta,$$

in which β and θ satisfy 3b, with $\beta' \geq 0$. Smooth β and

$$\theta = \beta - \int_{t_0}^{t} f\, d\xi :$$

$$\begin{aligned} \beta &= c_k + \int_{\tau_k}^{t} |f|\, d\xi \qquad (\tau_k \leq t \leq \tau_{k+1}); \\ \beta &= c_{k+1} = c_k + \int_{\tau_k}^{\tau_{k+1}} |f|\, d\xi \qquad (\tau_{k+1} \leq t \leq \tau_{k+2}), \end{aligned} \tag{11}$$

where c_k are constants, $f(\tau_k) = 0$, and the sequence τ_k is chosen so that 3b is satisfied for θ, determine such a decomposition.

3c. If

$$\int_{t_0}^{\infty} f\,d\xi = -\infty,$$

then a decomposition of type 3b, in which $\beta' \leq 0$ and $\beta \to -\infty$, is obtained by replacing $|f|$ in (11) by $(-|f|)$.

3d. If (5) is fulfilled, the change of variable

$$\tau = \int_{t_0}^{t} \exp\left(-2\int_{t_0}^{\eta} \theta\,d\xi\right) d\eta$$

transforms (1) into the equation

$$x''_{\tau} + f(\tau)x = 0; \qquad \int_0^{\tau} f\,d\xi = \int_{t_0}^{t} \exp\left[2\int_{t_0}^{\eta} \theta\,d\xi\right](\beta' + \theta^2)\,d\eta, \tag{12}$$

whose solutions are oscillatory or nonoscillatory simultaneously with the solutions of (1). Solving the question of the oscillation of (12), we obtain a second-degree condition for (1), which, by virtue of 3c, is freed from the requirement $\beta' + \theta^2 \geq 0$ present in (2) and absent in (6).

4. The solutions of (1) are oscillatory if:

4a. $\overline{\lim}\, I \neq \underline{\lim}\, I$ as $t \to \infty$, and there exists a $\beta \subset C^{(1)}$ (in particular, equal to the mean value of I) such that $\theta = \beta - I$ satisfies 3b (a generalization of ([7,8]) and others; the difficult case in ([13]), studied by the method of ([12]), see also ([14])).

4b. If $I \to +\infty$ as $t \to \infty$ (a generalization of ([5]) and many others).

4c. If $I \to -\infty$ as $t \to \infty$ and the construction 3c leads to fulfillment of 3b (under (9) or (10)—nonoscillatory).

4d. If $I = At + B + \varphi$, where A and B are constants, $\varphi \neq 0$ is an almost-periodic function whose mean value is equal to 0, and the mean value of $\varphi^2 + A \geq 0$ (if $\varphi^2 + A \leq 0$—nonoscillatory).

5. We obtain a partition of the set of equations (1) according to oscillation of their solutions by putting in (8) $\psi = g'/g - s$, where s is determined by 3a and $g > 0 \subset C^{(2)}$, and substituting β and θ in (5) and (6) (respectively, in (9) or in (10)). In particular, for $g \equiv t$, $g \equiv t^{\alpha}$, $g \equiv t \ln t$, etc., generalizations of Kneser' s conditions ([4,5]) and (8), as well as power and logarithmic conditions ([4,7,8]), etc., are obtained.

5a. If I tends to a finite limit as $t \to \infty$, then the question is decided by the function

$$\sigma = 4g \int_t^\infty \left(q - \frac{p^2}{4} - s'\right) d\xi - g',$$

obtained in item 5. In particular, for $g \equiv t$, $p \equiv s \equiv 0$, $q \geq 0$, $\sigma \geq \varepsilon > 0$, one obtains the Hille-Kneser condition ([6]). Here, too, conditions (5) and (6) or (9) considerably extend the domain of applications, including, for example, the result that equation (1), with $p \equiv 0$, $q = a \sin t/t$, has oscillatory solutions for $a > 1/\sqrt{2}$, and nonoscillatory solutions for $a \leq 1/\sqrt{2}$, etc.

6. If β_1 and θ_1 in the decomposition (4), constructed for I according to the rules of items 4 and 5, lead to the doubtful case, we construct conditions of the second (3d) and higher degrees. We obtain a sequence of transformations of equation (1) by putting in (12) $t = \tau_n$, $\tau = \tau_{n+1}$, $\theta = \theta_n$, and decomposing $\theta_n^2 = \lambda_n + \mu_n$ so that,

so that $\lambda_n > 0$ (in particular, equal to the mean value θ_n^2 (16)) and

$$\theta_{n+1} = -\int_{t_0}^t \mu_n \exp\left[2\int_{t_0}^\eta \theta_n \, d\xi\right] d\eta$$

ensured the simultaneous fulfillment of

$$\int_{t_0}^t \exp\left[-2\int_{t_0}^\eta \theta_{n+1} \, d\xi\right] d\eta = \infty \quad \text{and} \quad \int_{t_0}^\infty \exp\left[2\int_{t_0}^\eta \theta_{n+1} \, d\xi\right] d\eta = \infty.$$

Since $(\beta_{n+1})'_{\tau_{n+1}} = \lambda_n + (\beta_n)'_{\tau_n}$, the sequence $(\beta_n)'_{\tau_n}$ turns out to be increasing. For each of the pairs θ_n and β_n we write (6) and (10) and continue the process until one of these conditions is not fulfilled. In a special case the question is decided by the behavior of the limiting function for $(\beta_n)'_{\tau_n}$. The sequence of inverse substitutions would give an expansion (4) solving the problem. The use of the conditions in items 4 and 5 as higher-order conditions usually reduces the number of necessary operations.

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Note: Figure translations are in progress. See original paper for figures.