MATHEMATICS

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Abstract

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ON THE INTEGRAL REPRESENTATION OF LINEAR OPERATORS IN ORLICZ SPACES

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1. Let Z be a locally compact space; ν a positive Radon measure* on Z; $\mathcal{E} = (E(z))_{z\in Z}$ a family of Banach spaces and $\mathcal{E}' = (E'(z))_{z\in Z}$ a family of spaces dual to $E(z)$. Denote by $\mathcal{C}(\mathcal{E})$ (respectively by $\mathcal{C}(\mathcal{E}')$) the set of all vector fields** x (respectively functional fields x') defined on Z such that $x(z) \in E(z)$ (respectively $x'(z) \in E'(z)$) for every $z \in Z$.

We assume that there exists a family $\mathcal{A} \subset \mathcal{C}(\mathcal{E})$ of continuous vector fields and a family $\mathcal{A}' \subset \mathcal{C}(\mathcal{E}')$ of continuous functional fields, and that the following condition is satisfied:

For every $x \in \mathcal{A}$ and every $x' \in \mathcal{A}'$ the scalar function $z \to \langle x(z), x'(z)\rangle$ is continuous.

Let φ be a positive, increasing and left-continuous function defined on $[0, +\infty]$ such that $\varphi(0) = 0$ and $0 < \varphi(t) < +\infty$ for $0 < t < +\infty$; let ψ be the function "inverse" to φ; Φ and Ψ are the functions defined on $[0, +\infty]$ by the equalities

$$\Phi(u) = \int_0^u \varphi(t)\,dt, \qquad \Psi(v) = \int_0^v \psi(s)\,ds.$$

Consider the Orlicz space*** $\mathcal{L}^{\Phi}_{\mathcal{A}}(\nu)$. For each linear mapping f of the space $\mathcal{L}^{\Phi}_{\mathcal{A}}(\nu)$ into the Banach space F, put

$$|||f||| = \sup \sum_i |f(\varphi_{A_i} x_i)|, \qquad \|f\| = \sup \left|\sum_i f(\varphi_{A_i} x_i)\right|,$$

where the supremum is taken over all finite families $(A_i)_{1\le i\le n}$ of disjoint, relatively bicompact, Borel subsets of the space Z and over all finite families $(x_i)_{1\le i\le n}$ of elements of the basic family $\mathcal{A}$ such that $\left\|\sum_i \varphi_{A_i} x_i\right\|_{\Phi} \le 1$.

We have $\|f\| \le |||f||| \le +\infty$. If $F = C$, then $\|f\| = |||f|||$. If there exists a constant $M > 0$ such that $\Phi(2u) \le M\Phi(u)$ for $u > 0$, then

$$\|f\| = \sup |f(x)|, \quad x \in \mathcal{L}^{\Phi}_{\mathcal{A}}, \quad \|x\|_{\Phi} \le 1.$$

In ([4]) we proved the following theorem:

Theorem 1. *Let f be a linear mapping of $\mathcal{L}^{\Phi}_{A}$ into F. Suppose that: 1) there exists a constant $M > 0$ such that $\Phi(2u) \leq M\Phi(u)$ for $u > 0$; 2) $\mathcal{A}$ satisfies axiom (G)**; 3) F is a space dual to a separable Banach space S.*

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* For the theory of integration see ([1]).
** For the theory of vector fields see ([2]).
*** For the definition and properties of Orlicz spaces see ([3]).
**** (G). There exists a countable set $\mathcal{A}_0 \subset \mathcal{A}$ such that for every $z \in Z$ the set $\{x(z) \mid x \in \mathcal{A}_0\}$ is dense in $E(z)$.

We have $\||f\|| < +\infty$ if and only if there exists an operator field $z \to U(z) \in \mathcal{L}(E(z), F)$ such that $\|U\|_{\Psi} < +\infty$ and

$$\langle s, f(\mathbf{x})\rangle = \int \langle s, U(z)\mathbf{x}(z)\rangle \, d\nu(z), \qquad \mathbf{x} \in \mathcal{L}^{\Phi}_{A}, \quad s \in S. \tag{1}$$

In this case

$$^1/_2 \, \|U\|_{\Psi} \leq \||f\|| \leq \|U\|_{\Psi}. \tag{2}$$

In the present article, starting from a given operator field, we construct, without any assumptions concerning $\mathcal{A}$, F, and Φ, a linear mapping $f : \mathcal{L}^{\Phi}_{A} \to F$ satisfying equation (1), and seek a condition on $\mathcal{A}$ and F less restrictive than in the preceding theorem, such that f satisfy relation (2).

2. We first prove the following theorem:

Theorem 2. If $z \to U(z) \in \mathcal{L}(E(z), F)$ is an operator field such that, for every $\mathbf{x} \in \mathcal{L}^{\Phi}_{A}$ and every $y' \in F'$, the scalar function $z \to \langle U(z)\mathbf{x}(z), y'\rangle$ is ν-measurable and the function $z \to |U(z)\mathbf{x}(z)|$ is ν-integrable, then there exists a linear mapping $f : \mathcal{L}^{\Phi}_{A} \to F''$ such that

$$\langle f(\mathbf{x}), y'\rangle = \int \langle U(z)\mathbf{x}(z), y'\rangle \, d\nu(z), \qquad \mathbf{x} \in \mathcal{L}^{\Phi}_{A}, \quad y' \in F'; \tag{1'}$$

moreover, $\||f\|| \leq \|U\|_{\Psi} \leq +\infty$.

Proof. For $\mathbf{x} \in \mathcal{L}^{\Phi}_{A}$ and $y' \in F'$ the function $z \to \langle U(z)\mathbf{x}(z), y'\rangle$ is ν-integrable, since it is ν-measurable and

$$\int^* |\langle U(z)\mathbf{x}(z), y'\rangle| \, d\nu(z) \leq |y'| \int^* |U(z)\mathbf{x}(z)| \, d\nu(z) < +\infty.$$

The mapping

$$f(\mathbf{x}): \quad y' \to \int \langle U(z)\mathbf{x}(z), y' \rangle \, d\nu(z)$$

is linear and continuous:

$$\left| \int \langle U(z)\mathbf{x}(z), y' \rangle \, d\nu(z) \right| \le |y'| \int |U(z)\mathbf{x}(z)| \, d\nu(z).$$

Therefore $f(\mathbf{x}) \in F''$,

$$|f(\mathbf{x})| \le \int |U(z)\mathbf{x}(z)| \, d\nu(z),$$

$$\langle f(\mathbf{x}), y' \rangle = \int \langle U(z)\mathbf{x}(z), y' \rangle \, d\nu(z), \qquad \mathbf{x} \in \mathcal{L}_A^\Phi, \quad y' \in F'.$$

Obviously, the mapping f from the space $\mathcal{L}_A^\Phi$ into F'' is linear.

Let $\sum_{i=1}^n \varphi_{A_i} \mathbf{x}_i$ be a vector field such that the A_i are pairwise disjoint, relatively bicompact sets of the space Z; $\mathbf{x}_i$ are elements of the basic family $\mathcal{A}$, and

$$\left\| \sum_{i=1}^n \varphi_{A_i} \mathbf{x}_i \right\|_\Phi \le 1.$$

Then

$$\sum_{i=1}^n |f(\varphi_{A_i} \mathbf{x}_i)| \le \sum_{i=1}^n \int |U(z)\varphi_{A_i}(z)\mathbf{x}_i(z)| \, d\nu(z) =$$

$$= \int \left| U(z) \sum_{i=1}^n \varphi_{A_i}(z)\mathbf{x}_i(z) \right| d\nu(z) \le \|U\|_\Psi \left\| \sum_{i=1}^n \varphi_{A_i} \mathbf{x}_i \right\|_\Phi \le \|U\|_\Psi;$$

therefore $\||| f \|| \le \|U\|_\Psi$; this completes the proof.

3. We now seek sufficient conditions in order that f take values in F; that the given operator field U be unique, locally ν-almost everywhere determined by equation (1) or (1′); and that f satisfy relation (2).

Proposition 1. f takes values in F in each of the following cases:

a) F is dual to a Banach space S;

b) the operator field U is simply ν-measurable, i.e., for every $\mathbf{x} \in \mathcal{L}_\mathcal{A}^\Phi$ the function $z \to U(z)\mathbf{x}(z)$ (with values in F) is ν-measurable. In this case

$$f(\mathbf{x}) = \int U(z)\mathbf{x}(z)\,d\nu(z), \qquad \mathbf{x} \in \mathcal{L}^{\Phi}_{\mathcal{A}}; \tag{1"}$$

c) F is of countable type (in particular, $F = C$). In this case $(1'')$ holds.

To prove a), we consider $S \subset S'' \subset F'$ and, for the proof, take $y' \in S$.

For the proof of b), we note that the function $z \to U(z)\mathbf{x}(z)$ is ν-integrable for every $\mathbf{x} \in \mathcal{L}^{\Phi}_{\mathcal{A}}$. Then

$$\int \langle U(z)\mathbf{x}(z),\, y'\rangle\, d\nu(z) = \left\langle \int U(z)\mathbf{x}(z)\,d\nu(z),\, y' \right\rangle,$$

therefore

$$f(\mathbf{x}) = \int U(z)\mathbf{x}(z)\,d\nu(z).$$

c) is a consequence of b), since if F is of countable type, then U is simply ν-measurable.

Proposition 2. In each of the following cases the operator field U is uniquely, locally ν-almost everywhere, determined by the indicated equations:

a) $\mathcal{A}$ satisfies axiom (G) and there exists a countable subset $S \subset F'$ such that
$$|y| = \sup_{s\in S} \frac{|\langle y, s\rangle|}{|s|}$$
for every $y \in F$; equation $(1')$;

b) $\mathcal{A}$ satisfies axiom (G) and F is of countable type; equation $(1'')$;

c) $\mathcal{A}$ satisfies axiom (G) and F is dual to a Banach space S of countable type; equation (1);

d) $\mathcal{A}$ satisfies axiom (G) and U is simply ν-measurable; equation $(1'')$.

Cases b) and c) reduce to case a). Indeed, in case c) we may assume that $S \subset F'$; in case b), if (y_n) is a dense sequence in F, then for each n there exists an element $s_n \in F'$ such that $|s_n| = 1$ and $\langle y_n, s_n\rangle = |y_n|$; if we take $S = \{s_n\}$, then
$$|y| = \sup_n \frac{|\langle y, s_n\rangle|}{|s_n|}$$
for every $y \in F$.

To prove the uniqueness of U in case a), suppose that

$$\int \langle U(z)\mathbf{x}(z),\, y'\rangle\, d\nu(z) = 0$$

for every $\mathbf{x} \in \mathcal{L}^{\Phi}_{\mathcal{A}}$ and $y' \in F'$. There exists a ν-null set $N(\mathbf{x}, y')$ such that for $z \notin N(\mathbf{x}, y')$ we have $\langle U(z)\mathbf{x}(z), y' \rangle = 0$. The set

$$N(\mathbf{x}) = \bigcup_{y' \in S} N(\mathbf{x}, y')$$

is locally ν-null, and for $z \notin N(\mathbf{x})$ we have $U(z)\mathbf{x}(z) = 0$.

Let $(\mathbf{x}_n)$ be a sequence of elements of $\mathcal{A}$ such that for each $z \in Z$ the sequence $(\mathbf{x}_n(z))$ is dense in $E(z)$. For every bicompact set $K \subset T$, the set

$$N_K = \bigcup_{n=1}^{\infty} N(\mathbf{x}_n \varphi_K)$$

is ν-null, and for $z \notin N_K$ we have $U(z) = 0$. Hence it follows that ν-almost everywhere on K, $U(z) = 0$; therefore locally ν-almost everywhere $U(z) = 0$.

Proposition 3. The relations

$$^1/_2 \, \|U\|_\Psi \leq \|f\| \leq \|U\|_\Psi$$

hold in each of the following cases:

a) $\mathcal{A}$ satisfies axiom (G) and F is of countable type;

b) $F = C$ and the operator field $U = \mathbf{x}' \in \mathcal{C}(\mathcal{E}')$ is measurable with respect to $\mathcal{A}'$ and ν.

It remains for us to prove only the inequality $\frac{1}{2}\|U\|_\Psi \leq |||f|||$.

In case a), let (x_n) be a sequence dense in F. Construct a sequence (s_n) in F' such that $|s_n| = 1$ and $\langle x_n, s_n \rangle = |x_n|$ for every n. Let S be the closed subspace in F' generated by the sequence (s_n). Then S is of countable type, and F may be regarded as a subspace of the space S' dual to S; therefore, for every $z \in Z$, we may regard $U(z) \in \mathcal{L}(E(z), S')$. The inequality $\frac{1}{2}\|U\|_\Psi \leq |||f|||$ now follows from Theorem 1.

To prove b), we note that for $U = x'$ we have

$$\|x'\|_\Psi = \sup_{|x|_\Phi \leq 1} \int |\langle x(z), x'(z) \rangle| \, d\nu(z),$$

where $x \in O^{\Phi}_{\mathfrak{A}}$ (x is measurable with respect to $\mathfrak{A}$ and ν), and

$$|x|_\Phi = \int \Phi(|x(z)|) \, d\nu(z) < +\infty).$$

Since for every $x \in O^{\Phi}_{\mathfrak{A}}$ the function $z \to \langle x(z), x'(z) \rangle$ is integrable, we have

$$\|x'\|_\Psi = \sup_{|x|_\Phi \le 1} \left| \int \langle x(z), x'(z) \rangle \, d\nu(z) \right| =$$

$$= \sup_{|x|_\Phi \le 1} |f(x)| \le \sup_{|x|_\Phi \le 1} \|f\| \, \|x\|_\Phi \le \sup_{|x|_\Phi \le 1} \|f\|(|x|_\Phi + 1) \le 2\|f\| = 2|||f|||.$$

It follows from this that

$$\frac{1}{2}\|U\|_\Psi = \frac{1}{2}\|x'\|_\Psi \le |||f|||;$$

this completes the proof.

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Note: Figure translations are in progress. See original paper for figures.