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Abstract

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MATHEMATICS

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ON THE DECOMPOSITION OF Π-SOLVABLE GROUPS INTO A DIRECT PRODUCT OF SUBGROUPS

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§ 1. In papers ([4–6]) we introduced and, to a certain extent, investigated p-solvable and the more general Π-solvable groups, which subsequently also found application among other authors (P. Hall, G. Higman, B. Huppert, and others). In the present paper, which adjoins the note ([2]), a method is established for decomposing Π-solvable groups into a direct product of subgroups constructed with the aid of the concept of S-connectedness of prime divisors of the order of a group (cf. ([2])). At the same time Theorem 6 of ([2]) is generalized.

§ 2. We use the following definitions and notation (cf. ([1,2])). $\mathfrak{G}$ is a certain finite group of order $(\mathfrak{G})$; $\mathfrak{E}$ is the identity group; if n is a natural number, then $\Pi(n)$ is the set of all distinct prime divisors of n; Π is some empty or nonempty subset of the set $\Pi((\mathfrak{G}))$; a divisor d of the number $(\mathfrak{G})$ will be called a Π-divisor of $(\mathfrak{G})$ if $\Pi(d) \subseteq \Pi$; a Π-subgroup (Π-element) of the group $\mathfrak{G}$ is a subgroup (element) whose order is a Π-divisor of $(\mathfrak{G})$; a Π-subgroup whose order is the greatest Π-divisor of $(\mathfrak{G})$ will be called a Π-Sylow subgroup of $\mathfrak{G}$; $\mathfrak{G}_\Pi$ is the set of all Π-elements of the group $\mathfrak{G}$; p and q are prime numbers; a p-decomposable group is a finite group that is the direct product of its p-Sylow subgroup and a p-Sylow complement; $\mathfrak{P}$ is some p-Sylow subgroup of $\mathfrak{G}$; a special group is a finite group in which all Sylow subgroups are invariant; a group of type S is a finite nonspecial group all of whose nontrivial subgroups are special ([7]); the order of a group of type S has the form $p^\alpha q^\beta$, $\alpha > 0$, $\beta > 0$ ([8]); a symbol of the form $\mathfrak{S}_i(p,q) = \mathfrak{S}_i(q,p)$ will denote some subgroup of type S and of order of the form $p^\alpha q^\beta$; a pd-subgroup is a subgroup whose order is divisible by p.

Definition 1. If in $\mathfrak{G} \neq \mathfrak{E}$ there exists a sequence of subgroups of type S, which can be written in the form

$$\mathfrak{S}_1(p^{(1)}, p^{(2)}), \quad \mathfrak{S}_2(p^{(2)}, p^{(3)}), \quad \mathfrak{S}_3(p^{(3)}, p^{(4)}), \dots, \quad \mathfrak{S}_t(p^{(t)}, p^{(t+1)}),$$

where $p^{(1)} = p \in \Pi((\mathfrak{G}))$ and $p^{(t+1)} = q \in \Pi((\mathfrak{G}))$, then p and q will be called S-connected in the group $\mathfrak{G}$, and the indicated chain of subgroups will be called a chain S-connecting p and q.

Let now M be the set of all those $p \in \Pi((\mathfrak{G}))$ for which $\mathfrak{G}$ is not p-decomposable. With the aid of Theorem 3 of ([3]) it is not difficult to see that, if M is nonempty, then the property of S-connectedness divides M into classes $M_1, \dots, \dots, M_r$ of mutually S-connected numbers, which we shall call the S-classes of $\mathfrak{G}$.

Definition 2. A subset σ of the set $\Pi((\mathfrak{G}))$ will be called an S-portion of the group $\mathfrak{G}$ if σ consists of only one number p and $\mathfrak{G}$ is p-decomposable, or if σ is an S-class of $\mathfrak{G}$.

Theorem. Let $\mathfrak{G} \neq \mathfrak{E}$ be a Π-solvable group, let $\sigma_1, \sigma_2, \dots, \sigma_\mu$, $\mu \geq 0$, be all the S-portions of the group $\mathfrak{G}$ contained in Π, and let

τ is the union of all the remaining S-portions of $\mathfrak{G}$. Then:

1) $\mathfrak{G}_{\sigma_1}, \mathfrak{G}_{\sigma_2}, \dots, \mathfrak{G}_{\sigma_\mu}, \mathfrak{G}_\tau$ are subgroups and

$$\mathfrak{G} = \mathfrak{G}_{\sigma_1} \times \mathfrak{G}_{\sigma_2} \times \dots \times \mathfrak{G}_{\sigma_\mu} \times \mathfrak{G}_\tau;$$

2) the subgroup $\mathfrak{G}_{\sigma_i}$, $i = 1, 2, \dots, \mu$, is no longer decomposable into a direct product of nontrivial subgroups of pairwise coprime orders.

Proof. We shall first show that assertion 2) of the theorem follows from assertion 1).

Suppose the contrary: $\mathfrak{G}_{\sigma_i} = \mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2$, where $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ are relatively prime and greater than 1. Let $p \in \Pi((\mathfrak{H}_1))$ and $q \in \Pi((\mathfrak{H}_2))$. Since $\mathfrak{G}_{\sigma_i}$ is an invariant σ_i-Sylow subgroup of $\mathfrak{G}$ and σ_i is an S-portion of $\mathfrak{G}$, p and q are S-connected not only in $\mathfrak{G}$, but also in $\mathfrak{G}_{\sigma_i} = \mathfrak{H}$. Therefore, in the chain S-connecting p and q, there will be a subgroup $\mathfrak{S}_i(p^{(i)}, p^{(i+1)})$ of type S from $\mathfrak{H}$, for which $p^{(i)} \in \Pi((\mathfrak{H}_1))$ and $p^{(i+1)} \in \Pi((\mathfrak{H}_2))$. But then, since $((\mathfrak{H}_1), (\mathfrak{H}_2)) = 1$, the $p^{(i)}$-Sylow subgroup $\mathfrak{S}_i$ will lie in $\mathfrak{H}_1$, while the $p^{(i+1)}$-Sylow subgroup $\mathfrak{S}_i$ will lie in $\mathfrak{H}_2$. Hence, in view of $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2$, it follows that $\mathfrak{S}_i$ is a special group. A contradiction has been obtained.

Now suppose that $\mathfrak{G}$ is one of the groups of least order for which assertion 1) of the theorem fails. Since for Π empty $\mathfrak{G} = \mathfrak{G}_\tau$, Π is nonempty. Consequently, $(\mathfrak{G}) > 1$. Let then

$$\mathfrak{G} \supset \mathfrak{G}' \supset \dots$$

be some composition series of $\mathfrak{G}$.

Only the following two cases are possible:

1) The index of $\mathfrak{G}'$ in $\mathfrak{G}$ is not divisible by any prime from Π. Then τ is nonempty.

Since $(\mathfrak{G}') < (\mathfrak{G})$, for $\mathfrak{G}'$ there exists the required direct decomposition, which, as is not difficult to see, in the case under consideration may be given the form

$$\mathfrak{G}' = \mathfrak{G}'_{\sigma_1} \times \mathfrak{G}'_{\sigma_2} \times \ldots \times \mathfrak{G}'_{\sigma_\mu} \times \mathfrak{G}'_{\tau'}, \qquad \text{where } \tau' \subseteq \tau.$$

By Theorem 2 of ([6]), in $\mathfrak{G}$ there exist τ-Sylow subgroups. Let $\mathfrak{H}$ be one of them. Let now $p \in \tau$. Since $\mathfrak{G}_{\sigma_i}$, $1 \leq i \leq \mu$, is a characteristic subgroup of $\mathfrak{G}'$, $\mathfrak{G}_{\sigma_i}$ is invariant in $\mathfrak{G}$ and $\mathfrak{P}\mathfrak{G}_{\sigma_i}$ is a subgroup. Since p does not enter the S-portion σ_i of the group $\mathfrak{G}$, $\mathfrak{P}\mathfrak{G}_{\sigma_i}$ has no pd-subgroups of type S. Then, by Theorem 3 of paper ([3]), $\mathfrak{P}\mathfrak{G}_{\sigma_i}$ will be p-decomposable:

$$\mathfrak{P}\mathfrak{G}_{\sigma_i} = \mathfrak{P} \times \mathfrak{G}_{\sigma_i}.$$

Since $\mathfrak{P}$ is an arbitrary p-Sylow subgroup of $\mathfrak{G}$, in view of the fact that $\mathfrak{H}$ is a τ-Sylow subgroup of $\mathfrak{G}$, one may regard $\mathfrak{P}$ also as an arbitrary p-Sylow subgroup of $\mathfrak{H}$. Hence, taking into account

$$\mathfrak{P}\mathfrak{G}_{\sigma_i} = \mathfrak{P} \times \mathfrak{G}_{\sigma_i},$$

we see that

$$\mathfrak{H}\mathfrak{G}_{\sigma_i} = \mathfrak{H} \times \mathfrak{G}_{\sigma_i}, \qquad 1 \leq i \leq \mu.$$

But it is obvious that

$$\mathfrak{G} = (\mathfrak{G}_{\sigma_1} \times \mathfrak{G}_{\sigma_2} \times \ldots \times \mathfrak{G}_{\sigma_\mu} \times \mathfrak{G}_{\tau'})\mathfrak{H}.$$

Hence, taking the preceding equalities into account, we see that

$$\mathfrak{G} = \mathfrak{G}_{\sigma_1} \times \mathfrak{G}_{\sigma_2} \times \ldots \times \mathfrak{G}_{\sigma_\mu} \times (\mathfrak{G}_{\tau'}\mathfrak{H}),$$

whence

$$\mathfrak{G}_{\tau'}\mathfrak{H} = \mathfrak{H} = \mathfrak{G}_{\tau}.$$

A contradiction has been obtained.

2) The index of $\mathfrak{G}'$ in $\mathfrak{G}$ is equal to $p \in \Pi$.

Since $\mathfrak{G}'$ is also Π-solvable, and $(\mathfrak{G}') < (\mathfrak{G})$, for $\mathfrak{G}'$ there exists the decomposition required by the theorem:

$$\mathfrak{G}' = \mathfrak{G}'_{\sigma'_1} \times \mathfrak{G}'_{\sigma'_2} \times \ldots \times \mathfrak{G}'_{\sigma'_{\mu'}} \times \mathfrak{G}'_{\tau''}, \qquad \mu' \geq 0, \quad \tau' \subseteq \tau,$$

and each $\sigma'_i \subseteq \Pi$, $i = 1, 2, \ldots, \mu'$, is contained in some S-portion of $\mathfrak{G}$. Then

$$\mathfrak{G} = \mathfrak{P}\mathfrak{G}'.$$

Further, two cases are possible.

a) $\mathfrak{G}$ is p-decomposable. If $\mathfrak{G}'$ is not a pd-subgroup, then

$$\mathfrak{G} = \mathfrak{P} \times \mathfrak{G}'_{\sigma'_1} \times \mathfrak{G}'_{\sigma'_2} \times \ldots \times \mathfrak{G}'_{\sigma'_{\mu'}} \times \mathfrak{G}'_{\tau'}$$

will obviously be the desired decomposition of $\mathfrak{G}$. Let $\mathfrak{G}'$ be a pd-group. Since $\mathfrak{G}'$ is also p-decomposable, one of the sets $\sigma'_1, \sigma'_2, \ldots, \sigma'_{\mu'}$ (for example, σ'_1) will be of the form $\{p\}$. Then

$$\mathfrak{G} = \mathfrak{P} \times \mathfrak{G}'_{\sigma'_2} \times \ldots \times \mathfrak{G}'_{\sigma'_{\mu'}} \times \mathfrak{G}'_{\tau'}$$

will obviously be a decomposition of the required form. A contradiction has been obtained.

b) $\mathfrak{G}$ is not p-decomposable. Then, by Theorem 3 of ([3]), the number p is S-connected with some elements of $\Pi((\mathfrak{G}'))$. Let ρ be the union of all those

sets from the collection $\sigma'_1, \sigma'_2, \ldots, \sigma'_\mu, \tau'$, which contain at least one element S-connected with ρ. In the present case ρ is nonempty.

It is clear that $\mathfrak{G}'_\rho$ will be the product of the elements of some nonempty collection $\mathfrak{M}$ of direct factors of the decomposition

$$\mathfrak{G}' = \mathfrak{G}'_{\sigma'_1} \times \mathfrak{G}'_{\sigma'_2} \times \cdots \times \mathfrak{G}'_{\sigma'_\mu} \times \mathfrak{G}'_{\tau'}.$$

If outside $\mathfrak{M}$ there are no longer any factors of this decomposition, then $\mathfrak{G} = \mathfrak{P}\mathfrak{G}'_\rho$. It is clear that then $\Pi((\mathfrak{P}\mathfrak{G}'_\rho))$ will be either the union τ of the S-portions of the group $\mathfrak{G}$ not entering into Π, or the S-portion $\sigma = \Pi$ of the group $\mathfrak{G}$. Then either $\mathfrak{G} = \mathfrak{G}_\tau$, or $\mathfrak{G} = \mathfrak{G}_\sigma$. We have obtained a contradiction.

Let outside $\mathfrak{M}$ there exist factors $\mathfrak{G}'_{\omega_1}, \mathfrak{G}'_{\omega_2}, \ldots, \mathfrak{G}'_{\omega_\lambda}$ of the above decomposition of $\mathfrak{G}'$ (clearly, $\omega_1, \omega_2, \ldots, \omega_\lambda$ coincide with some of the sets $\sigma'_1, \sigma'_2, \ldots, \sigma'_\mu, \tau'$).

Since $\omega_1, \omega_2, \ldots, \omega_\lambda$ have no elements in common with the set ρ, as in case 1), we are convinced that

$$\mathfrak{P}\mathfrak{G}'_{\omega_i} = \mathfrak{P} \times \mathfrak{G}'_{\omega_i}, \quad i = 1, 2, \ldots, \lambda.$$

Then

$$\mathfrak{G} = \mathfrak{P}\mathfrak{G}' = (\mathfrak{P}\mathfrak{G}'_\rho) \times \mathfrak{G}'_{\omega_1} \times \mathfrak{G}'_{\omega_2} \times \cdots \times \mathfrak{G}'_{\omega_\lambda}.$$

If τ' is nonempty and enters into ρ, then put $\tau = \Pi((\mathfrak{P}\mathfrak{G}'_\rho))$; then all ω_i, $i = 1, 2, \ldots, \lambda$, will be S-portions of $\mathfrak{G}$ entering into Π.

If, however, τ' is nonempty and coincides with one of the sets $\omega_1, \omega_2, \ldots, \omega_\lambda$ (or is empty), then the remaining ones among them (or all of them), as well as $\Pi((\mathfrak{P}\mathfrak{G}'_\rho))$, will clearly be S-portions of $\mathfrak{G}$ entering into Π. In all cases the decomposition of $\mathfrak{G}$ obtained will be of the required type. A contradiction has been obtained. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.