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Abstract

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MATHEMATICS

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ON THE MULTIPLICITY OF THE SPECTRUM OF A SECOND-ORDER DIFFERENTIAL OPERATOR

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In Sec. 1 we give some results concerning the multiplicity of the operator T_S of multiplication by the independent variable in the space \mathcal{L}_S^2 , generated (see (2)) by a nondecreasing matrix function $S(t) = \|\sigma_{ij}(t)\|_{i,j=1}^2$, constructed in a special way. In Secs. 2 and 3 these results are applied to differential operators.

1. We assign a function $\theta(\lambda)$ of a complex variable to the R -functions if it is defined and holomorphic in each of the half-planes $\text{Im } \lambda > 0$ and $\text{Im } \lambda < 0$, and has the following properties: a) $\theta(\bar{\lambda}) = \overline{\theta(\lambda)}$, b) $\text{Im } \lambda \text{ Im } \theta(\lambda) \geq 0$ ($\text{Im } \lambda \neq 0$).

As is known, every R -function $\theta(\lambda)$ admits a unique representation of the form

$$\theta(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\rho(t) \quad (\text{Im } \lambda \neq 0), \quad (1)$$

where $\beta > 0$ and α are real constants, and $\rho(t)$ is a nondecreasing function such that $\rho(0) = 0$, $\rho(t) = \frac{1}{2}[\rho(t-0) + \rho(t+0)]$, and the integral

$$\int_{-\infty}^{\infty} (1+t^2)^{-1} d\rho(t). \quad (2)$$

converges. The function $\rho(t)$ is called the spectral function of the R -function $\theta(\lambda)$.

Let $\omega_l(\lambda)$ and $\omega_r(\lambda)$ be two R -functions, and let $\tau_l(t)$ and $\tau_r(t)$ be their spectral functions. Define the functions $\Omega_{ij}(\lambda)$ ($i, j = 1, 2$) by the equalities

$$\begin{aligned} \Omega_{11}(\lambda) &= \omega_l(\lambda)\omega_r(\lambda)(\omega_l(\lambda) + \omega_r(\lambda))^{-1}, \\ \Omega_{22}(\lambda) &= -(\omega_l(\lambda) + \omega_r(\lambda))^{-1}, \\ \Omega_{12}(\lambda) &= \Omega_{21}(\lambda) = \omega_l(\lambda)(\omega_l(\lambda) + \omega_r(\lambda))^{-1} \end{aligned} \quad (\text{Im } \lambda \neq 0) \quad (3)$$

and the functions $\sigma_{ij}(t)$ by the equalities

$$\sigma_{ij}(t) = \lim_{\eta \rightarrow +0} \frac{1}{\pi} \int_0^t \operatorname{Im} \Omega_{ij}(s + i\eta) ds \quad (-\infty < t < \infty; i, j = 1, 2). \quad (4)$$

It is easy to verify directly that, for any real ξ_1 and ξ_2 , the function $\Phi_\xi(\lambda) = \sum \xi_i \xi_j \Omega_{ij}(\lambda)$ is an R -function. Hence it follows that, for any real ξ_1 and ξ_2 , the function

$$\sum_{i,j=1}^2 \xi_i \xi_j \sigma_{ij}(t)$$

is nondecreasing in t , i.e. the matrix function $S(t) = \|\sigma_{ij}(t)\|_1^2$ is nondecreasing (or, as is accepted in ^(1,4), a matrix-valued distribution function*).

Let $\sigma(t) = \sigma_{11}(t) + \sigma_{22}(t)$ ($-\infty < t < \infty$). It is easy to see that all the functions $\sigma_{ij}(t)$ ($i, j = 1, 2$) are σ -absolutely continuous and, consequently,

* In ^(1,4) it was required of the matrix $S(t)$ that it be left-continuous, whereas in the present work, by virtue

σ -almost everywhere on $(-\infty, \infty)$ there exist derivatives $\delta_{ij}(t) = d\sigma_{ij}(t)/d\sigma(t)$. The set of points $t \in (-\infty, \infty)$ at which all elements of the matrix $\delta(t) = \|\delta_{ij}(t)\|_1^2$ are defined will be denoted by Q .

Consider the Hilbert space \mathcal{L}_S^2 of vector-functions generated by the matrix $S(t)$. This space consists of all σ -measurable vector-functions $f(t) = \{f_1(t), f_2(t)\}$ ($-\infty < t < \infty$) for which

$$\|f\|^2 = (f, f) = \int_{-\infty}^{\infty} \left(\sum_{i,j=1}^2 f_i(t) \overline{f_j(t)} \delta_{ij}(t) \right) d\sigma(t) < \infty.$$

In \mathcal{L}_S^2 the scalar product is defined by the equality

$$(f, g) = \int_{-\infty}^{\infty} \left(\sum_{i,j=1}^2 f_i(t) \overline{g_j(t)} \delta_{ji}(t) \right) d\sigma(t) \quad *. \quad (5)$$

Let T_S be the operator of multiplication by the independent variable t in the space \mathcal{L}_S^2 . This is, as is known, a self-adjoint operator. The main theorem formulated below concerns the multiplicity of the spectrum of the operator T_S .

Introduce notation. Let $\rho(t)$ ($\rho(0) = 0$; $\rho(t) = \frac{1}{2}(\rho(t-0) + \rho(t+0))$) be a nondecreasing function on $(-\infty, \infty)$ such that the integral (2) converges. Denote by $E_a[\rho]$ the set of those points $t \in (-\infty, \infty)$ at which there exists a finite derivative $\rho'(t)$, different from zero, and by $E_{a+}[\rho]$ the set of those points $t \in$

$E_a[\rho]$ at which there exists the finite limit $\lim_{\eta \rightarrow +0} \theta(t + i\eta)$, where $\theta(\lambda)$ is any R -function whose spectral function is $\rho(t)$ (all such R -functions, as follows from (1), differ only by a term linear with respect to λ). It is known that the Lebesgue measure of the set $E_a[\rho] \setminus E_{a+}[\rho]$ is equal to zero.

Main theorem. *If the Lebesgue measure of the set $M = E_a[\tau_l] \cap E_a[\tau_r]$ is equal to zero, then the operator T_S has simple spectrum; if the Lebesgue measure of the set M is positive, then the multiplicity of the spectrum of the operator T_S is equal to two, and in the second case the multiplicity of the covering** of the set $M_+ = E_{a+}[\tau_l] \cap E_{a+}[\tau_r]$ is equal to two, while the multiplicity of the covering of its complement $CM_+ = (-\infty, \infty) \setminus M_+$ does not exceed one.*

Corollary. *If the function $\tau_l(t)$ (or $\tau_r(t)$) is singular**, then the operator T_S has simple spectrum independently of the behavior of the function $\tau_r(t)$ ($\tau_l(t)$).**

Let us explain the idea of the proof. With the aid of theorems on the behavior of R -functions on the boundary of the upper half-plane it is proved that at the points of the set $M_+ \cap Q$ the rank of the matrix $\delta(t) = \|\delta_{ij}(t)\|$ is equal to two, while on the set $Q \setminus M_+$ the rank of this matrix is σ -almost everywhere equal to one. In addition, it is proved that the σ -measure of the set M_+ is positive if and only if its Lebesgue measure is positive and, consequently, the Lebesgue measure of the set M is positive. These properties of the matrix $\delta(t)$ are equivalent to the assertion of the main theorem (see in this regard (2), pp. 109–110).

Let us dwell on the case when, for example, $\tau_l(t)$ is a singular function. Let N_1 be the set of those points $t \in Q$ at which there exists a real

* The definition of the space \mathcal{L}_S^2 given in (2) (see also (1), p. 281) is equivalent, as was shown also in (2), to the one given here. Note that in works (1, 4, 6) the integral appearing on the right-hand side of equality (5) is conventionally written in the form

$$\int_{-\infty}^{\infty} \sum_{i,j=1}^2 f_i(t) \overline{g_j(t)} d\sigma_{ij}(t).$$

** We adhere to the definition of spectral multiplicity given in (1) (p. 278), and to the definition of covering multiplicity given in (5) (p. 160).

***) A nondecreasing function $\rho(t)$ on $(-\infty, \infty)$ is called singular if its derivative $\rho'(t)$ is almost everywhere equal to zero. A singular function is representable as the sum of a jump function and a continuous singular function.

the limit

$$\omega_l(t) = \lim_{\eta \rightarrow +0} \omega_l(t + i\eta),$$

and $N_2 = Q \setminus N_1$. It turns out that, σ -almost everywhere on N_1 ,

$$\delta_{12}(t) = \delta_{21}(t) = -\delta_{22}(t)\omega_l(t), \quad \delta_{11}(t) = \delta_{22}(t)\omega_l^2(t) \quad (6)$$

and σ -almost everywhere on N_2

$$\delta_{11}(t) = 1, \quad \delta_{12}(t) = \delta_{21}(t) = \delta_{22}(t) = 0. \quad (7)$$

2. Consider the quasi-differential expression*

$$l[y] = -\frac{d}{dx} \left(p(x) \frac{dy(x)}{dx} \right) + q(x)y(x) \quad (a < x < b), \quad (8)$$

where $-\infty \leq a < b \leq \infty$, and $p^{-1}(x)$ and $q(x)$ are real measurable functions summable on every segment $[a_1, b_1] \subset (a, b)$. With the aid of $l[y]$ we define, as was done in ⁽⁴⁾, §17, a differential operator \mathcal{L}_0 in the Hilbert space $H = \mathcal{L}^2(a, b)$. This operator is symmetric and has equal defect numbers not exceeding two. In this section we shall consider only the case when the operator \mathcal{L}_0 is self-adjoint.**

Let c be an arbitrary point of (a, b) . Denote by $u_1(x, \lambda)$ and $u_2(x, \lambda)$ the solutions of the equation $l[y] - \lambda y = 0$ satisfying the conditions

$$u_1(c, \lambda) = 1, \quad p(x)u_1'(x, \lambda)|_{x=c} = 0; \quad u_2(c, \lambda) = 0, \quad p(x)u_2'(x, \lambda)|_{x=c} = -1.$$

As is known, for every non-real λ there are uniquely determined numbers $\omega_l(\lambda)$ and $\omega_r(\lambda)$ such that

$$u_2(x, \lambda) - \omega_l(\lambda)u_1(x, \lambda) \in \mathcal{L}^2(a, c); \quad u_2(x, \lambda) + \omega_r(\lambda)u_1(x, \lambda) \in \mathcal{L}^2(c, b). \quad (9)$$

The functions $\omega_l(\lambda)$ and $\omega_r(\lambda)$ are R -functions. Their spectral functions $\tau_l(\lambda)$ and $\tau_r(\lambda)$ are the spectral functions of the differential systems

$$l[y] - \lambda y = 0 \quad (a < x \leq c), \quad y(c) = 1, \quad p(x)y'(x)|_{x=c} = 0; \quad (10)$$

$$l[y] - \lambda y = 0 \quad (c \leq x < b), \quad y(c) = 1, \quad p(x)y'(x)|_{x=c} = 0 \quad (11)$$

respectively (in the case under consideration each of these differential systems has a unique spectral function).

It is also known that the operator \mathcal{L}_0 is unitarily equivalent to the operator T_S of multiplication by the independent variable in the space \mathcal{L}_S^2 , where $S(t) = \|\sigma_{ij}(t)\|_1^2$ is the matrix function defined by the equalities (3)-(4), starting from the R -functions $\omega_l(\lambda)$ and $\omega_r(\lambda)$ satisfying conditions (9). In this connection the following proposition follows from the main theorem:

Theorem. Let $\tau_l(\lambda)$ and $\tau_r(\lambda)$ be the spectral functions of the differential systems (10) and (11), respectively. Then, if the Lebesgue measure of the set $M = E_a[\tau_l] \cap E_a[\tau_r]$ is zero, the operator \mathcal{L}_0 has simple spectrum; if $m(M) > 0$, then the multiplicity of the spectrum of the operator \mathcal{L}_0 is equal to two; moreover, in the latter case the multiplicity of the covering of the set $M_+ = E_{a+}[\tau_l] \cap E_{a+}[\tau_r]$ is equal to two, and that of its complement CM_+ does not exceed one.

Corollary. If the spectral function of the differential system (10) (of the differential system (11)) is singular, then the operator \mathcal{L}_0 has simple spectrum independently of the behavior of the functions $p(x)$ and $q(x)$ in a neighborhood of the point $x = b$ ($x = a$).

* Everything set forth in this section carries over to more general differential expressions (see, for example, those considered in (3)).

** In a joint discussion of A. V. Shtraus' work (6), M. G. Krein and the author came to the conclusion that, in the case of defect index (1, 1), all self-adjoint extensions of the operator \mathcal{L}_0 have simple spectrum, while in the case of defect index (2, 2) simple spectrum is possessed by all self-adjoint extensions of the operator \mathcal{L}_0 with separated boundary conditions, and, moreover, that in the case of defect index (0, 0), for simplicity of the spectrum of the operator \mathcal{L}_0 it is enough that the spectrum of at least one of the differential systems (10) and (11) be discrete.

In the case when the operator \mathcal{L}_0 has a simple spectrum, define for each $t \in Q$ the function

$$u(x, t) = \text{sign } \delta_{12}(t) \cdot \sqrt{\delta_{11}(t)} u_1(x, t) + \sqrt{\delta_{22}(t)} u_2(x, t) \quad (a < x < b),$$

which is, naturally, a solution of the equation $l[y] - ty = 0$. It turns out that in this case, for every function $f(x) \in \mathcal{L}^2(a, b)$, the inversion formulas

$$F(t) = \int_a^b f(x) u(x, t) dx, \quad f(x) = \int_{-\infty}^{\infty} F(t) u(x, t) d\sigma(t), \quad (12)$$

hold, where the integrals converge in the spaces $\mathcal{L}_\sigma^2(-\infty, \infty)$ and $\mathcal{L}^2(a, b)$, respectively, and

$$\int_{-\infty}^{\infty} |F(t)|^2 d\sigma(t) = \int_a^b |f(x)|^2 dx. \quad (13)$$

Remark. It is not difficult to verify that the sets $E_a[\tau]$, $E_{a+}[\tau]$ and $E_a[\tau_f]$, $E_{a+}[\tau_f]$ do not depend on the position of the point c and, consequently, depend only on the behavior of the coefficients $p(x)$ and $q(x)$ in neighborhoods of the points $x = a$ and $x = b$, respectively.

3. Let now the left endpoint $x = a$ of the differential expression (8) be regular, i.e. $a > -\infty$ and the functions $p^{-1}(x)$ and $q(x)$ are summable in a right neighborhood of the point $x = a$, while at the endpoint $x = b$ the Weyl point case occurs (under these conditions the operator \mathcal{L}_0 has defect index $(1, 1)$). Consider the boundary-value problem

$$l[y] - \lambda y = 0 \quad (a \leq x < b), \quad y(a) = \Omega(\lambda) p(x) y'(x) \Big|_{x=a}, \quad (14)$$

where $\Omega(\lambda)$ is an R -function. A. V. Shtraus ⁽⁶⁾ showed that, in the case when the R -function $\Omega(\lambda)$ is meromorphic,* there exists a nondecreasing function $\sigma(t)$ ($-\infty < t < \infty$) such that, for every function $f(x) \in \mathcal{L}^2(a, b)$, the equalities (12), (13) hold, where $u(x, \lambda)$ is a solution of the boundary-value problem (14) (⁽⁶⁾, Theorem 1).

Theorem 1 from ⁽⁶⁾ (in its complete formulation) generalizes to the case when the spectral function of the R -function $\Omega(\lambda)$ is singular; only in the boundary condition of the boundary-value problem (14), for those real λ for which there exists a finite real limit $\lim_{\eta \rightarrow +0} \Omega(\lambda + i\eta)$ (this holds almost everywhere on $(-\infty, \infty)$), one must take $\Omega(\lambda)$ equal to this limit, while for the remaining real λ one must take $\Omega(\lambda) = \infty$, i.e. understand the boundary condition in (14) as $p(x) y'(x) \Big|_{x=a} = 0$.

This generalization of Theorem 1 from ⁽⁶⁾ follows easily from the corollary of the main theorem, equalities (6), (7), and the formulas given in ⁽⁶⁾ on p. 785. The main theorem shows that no further generalization of Theorem 1 from ⁽⁶⁾ in this direction is possible.

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* For meromorphic R -functions the spectral function is a step function, and the set of jump points coincides with the set of poles of the R -function.

Note: Figure translations are in progress. See original paper for figures.

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