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**Abstract**

**Full Text**

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**MATHEMATICS**

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### **IDENTITIES IN SPECIAL SEMIGROUPS**

*(Presented by Academician P. S. Novikov on November 13, 1961)*

We shall call an associative system (semigroup)  $\Pi$  special if it can be defined by means of a system of defining relations of the form

$$A = \Lambda, \tag{1}$$

where the symbol  $\Lambda$  denotes the empty word.

The purpose of the present note is to investigate the question of the possibility of identity relations in special semigroups that are not groups. The free semigroup  $\Pi_0$  with one generator is defined, in particular, by the empty set of defining relations of the form (1). By  $\Pi_1$  we shall denote the semigroup defined by two generators  $a$  and  $b$  and one defining relation  $ab = \Lambda$ .

**Theorem 1.** *Let the special semigroup  $\Pi$  be isomorphic neither to the semigroup  $\Pi_0$  nor to  $\Pi_1$ . If some identity relation holds in  $\Pi$ , then  $\Pi$  is a group.*

In what follows, by  $\Pi$  we shall denote a special semigroup defined by some alphabet  $\mathfrak{A}$  and some system of defining relations of the form (1). A word  $A$  in the alphabet  $\mathfrak{A}$  will be called right (left) invertible in the semigroup  $\Pi$  if one can find such a word  $X$  in the alphabet  $\mathfrak{A}$  that  $AX = \Lambda$  ( $XA = \Lambda$ ) in the semigroup  $\Pi$ . If the word  $A$  is invertible both on the left and on the right, then we shall call it invertible. The following lemma is obvious.

**Lemma 1.** *If  $AB$  and  $BC$  are invertible in  $\Pi$ , then  $A$ ,  $B$ , and  $C$  are also invertible in  $\Pi$ .*

We shall denote the graphical equality of words by the sign  $\doteq$ . A set  $M$  of words in the alphabet of the semigroup  $\Pi$  will be called **free relative to  $\Pi$**  if, for any words  $A, B \in M$ , from  $A = B$  in  $\Pi$  it follows that  $A \doteq B$ . Let  $a$  and  $b$  be generators of the semigroup  $\Pi$ , and let  $\alpha$  be a natural number. By  $M_\alpha^\Pi$  we shall denote the set of all subwords invertible in the semigroup  $\Pi$  of words of the form

$$b^{r_1}ab^\alpha b^{r_2}ab^\alpha b^{r_3} \dots b^r tab^\alpha b^{r_{t+1}}, \quad (2)$$

and by  $N_\alpha^\Pi$  the set of all subwords invertible in  $\Pi$  of words of the form

$$a^{r_1}a^\alpha ba^{r_2}a^\alpha ba^{r_3} \dots a^r ta^\alpha ba^{r_{t+1}}, \quad (3)$$

**Lemma 2.** *If  $a$  and  $b$  are letters of the alphabet  $\mathfrak{A}$  and, for some natural number  $\alpha$ , the set  $M_\alpha^\Pi$  ( $N_\alpha^\Pi$ ) is free relative to the semigroup  $\Pi$ , then no identity relation holds in  $\Pi$ .*

We omit the proof of Lemma 2. We shall only show how Theorem 1 follows from this lemma.

**Lemma 3.** *If the alphabet  $\mathfrak{A}$  of the semigroup  $\Pi$  contains two letters  $a$  and  $b$  that are not left (right) invertible in  $\Pi$ , then no identity relation holds in  $\Pi$ .*

**Proof.** Indeed, if  $a$  and  $b$  are not left-invertible in  $\Pi$ , then no invertible word in  $\Pi$  can end with the letter  $a$  or  $b$ . In this case the set  $M_\alpha^\Pi$ , for any  $\alpha$ , contains only the empty word, since no nonempty subword of words of the form (2) will be invertible in  $\Pi$ . Then the set  $M_\alpha^\Pi$  is free with respect to  $\Pi$ . The same will hold if  $a$  and  $b$  are not right-invertible. It remains to apply Lemma 2.

**Lemma 4.** *If a semigroup  $\Pi$  contains an invertible generator  $a$ , not equal to  $\Lambda$  in  $\Pi$ , and a generator  $b$  that is not left-invertible (right-invertible), then no identical relation holds in  $\Pi$ .*

**Proof.** We shall consider in detail only the case in which the letter  $b$  is not left-invertible. The other case is considered analogously. Let  $E$  be a subword, invertible in  $\Pi$ , of some word of the form (2) with  $\alpha = 1$ . If  $E$  is nonempty, then  $E \doteq E_1 a$ , since, by assumption, the letter  $b$  is not left-invertible. From the invertibility of  $E_1 a$  and  $a$ , by Lemma 1 it follows that  $E_1$  is invertible. Then  $E_1$  cannot end with the letter  $b$ , since  $b$  is not left-invertible. But  $E_1$  also cannot end with the letter  $a$ , since two letters  $a$  cannot stand next to each other in  $E$ . Consequently,  $E_1$  is empty. We have proved that if an arbitrary invertible subword  $E$  of a word of the form (2) with  $\alpha = 1$  is nonempty, then  $E \doteq a$ . Thus the set  $M_1^\Pi$  in our case contains two elements:  $\Lambda$  and  $a$ . Since, by the hypothesis of the lemma,  $a \neq \Lambda$  in  $\Pi$ , the set  $M_1^\Pi$  is free in  $\Pi$ .

**Lemma 5.** *Let the semigroup  $\Pi$  contain only two generators  $a$  and  $b$ , not equal to  $\Lambda$  in  $\Pi$ . If, moreover, the word  $a^\gamma b$  is not invertible in  $\Pi$  for every natural  $\gamma$ , and for  $\gamma = \gamma_1 > 0$  is right-invertible, then no identical relation holds in  $\Pi$ .*

**Proof.** By assumption,  $a^{\gamma_1} b$  is right-invertible in the semigroup  $\Pi$ , i.e., there exists such an  $X$  that  $a^{\gamma_1} b X = \Lambda$  in  $\Pi$ , and it may be assumed that the word  $X$  contains only the generators  $a$  and  $b$ . From  $\gamma_1 > 0$  it follows that the letter  $a$  is right-invertible. If the letter  $a$  were also left-invertible, then by Lemma 1 the word  $a^{\gamma_1} b$  would be invertible in  $\Pi$ . Consequently, the letter  $a$  is not left-invertible in  $\Pi$ . We shall show that then the set  $N_{\gamma_1}^\Pi$  of subwords, invertible in

$\Pi$ , of words of the form (3) with  $\alpha = \gamma_1$ , is free with respect to  $\Pi$ . Let  $E$  be a subword, invertible in  $\Pi$ , of some word of the form (3) with  $\alpha = \gamma_1$ . If  $E$  is nonempty, then  $E \doteq E_1 b$ , since we have established that  $a$  is not left-invertible. But, by the hypothesis of the lemma, no word  $a^\gamma b$  is invertible. Consequently,  $E \doteq E_2 b a^i a^{\gamma_1} b$ . Then the word  $a^{\gamma_1} b$  is left-invertible. This contradicts the hypothesis of the lemma, since, by assumption,  $a^{\gamma_1} b$  is right-invertible and not invertible in  $\Pi$ . Hence the word  $E$  is empty, i.e., the set  $N_{\gamma_1}^\Pi$  contains the unique element  $\Lambda$ . Lemma 5 is proved.

**Lemma 6.** *Let the semigroup  $\Pi$  not be a group and contain only two generators  $a$  and  $b$ , not equal to  $\Lambda$  in  $\Pi$ . If, moreover, the word  $a^\gamma b$  is invertible in  $\Pi$  for some  $\gamma > 1$ , then no identical relation holds in  $\Pi$ .*

**Proof.** Since, by assumption,  $\Pi$  is not a group, from the invertibility of  $a^\gamma b$  it follows that the letter  $a$  is right-invertible but not left-invertible, while  $b$  is left-invertible but not right-invertible. We shall show that the set  $M_1^\Pi$  is free with respect to  $\Pi$ . Let  $E$  be an invertible subword of some word of the form (2) with  $\alpha = 1$ . If  $E$  is nonempty, then  $E \doteq a E_1$ , for  $b$  is not right-invertible. Since  $a$  is not left-invertible,  $E_1$  is nonempty, and  $E_1 \doteq b E_2$ , for two letters  $a$  cannot stand next to each other in  $E$ . Thus, if  $E$  is nonempty, then  $E \doteq a b E_2$ . From the invertibility of  $a^\gamma b$  and  $a b E_2$ , by Lemma 1 it follows that  $a^{\gamma-1}$  is invertible. But this is impossible, since, by assumption,  $\gamma > 1$ , while the letter  $a$  is not left-invertible. Consequently,  $E$  is empty, i.e., the only element of the set  $M_1^\Pi$  is the empty word  $\Lambda$ . It remains to apply Lemma 2.

**Lemma 7.** *If the semigroup  $\Pi$  is not a group and contains only two generators  $a$  and  $b$ , which are not equal to  $\Lambda$  in it, and the word  $ab$  also*

*is not equal to  $\Lambda$ , but is invertible in  $\Pi$ , then no identity relation holds in  $\Pi$ .*

**Proof.** It is enough to prove that the set  $M_\alpha^\Pi$  for  $\alpha = 2$  is free relative to  $\Pi$ . From the invertibility of  $ab$  in  $\Pi$  it follows that  $a$  is right-invertible but not left-invertible, and  $b$  is left-invertible but not right-invertible. Let  $E \in M_2^\Pi$ . Then either  $E \doteq \Lambda$ , or  $E \doteq a E_1$ , with  $E_1$  nonempty. Since  $E$  is a subword of some word of the form (2) for  $\alpha = 2$ ,  $E \doteq a b E_2$ . From the invertibility of the words  $ab$  and  $a b E_2$ , by Lemma 1 it follows that the word  $E_2$  is invertible. If  $E_2$  is nonempty, then  $E_2 \doteq b E_3$ . But this is impossible, since then  $b$  would be right-invertible. Hence either  $E \doteq \Lambda$ , or  $E \doteq a b$ . Since, by hypothesis,  $ab \neq \Lambda$  in  $\Pi$ ,  $M_2^\Pi$  is free relative to  $\Pi$ .

**Lemma 8.** *Let the semigroup  $\Pi$  not be a group and contain only two generators  $a$  and  $b$ , which are not equal to  $\Lambda$  in it. If, moreover,  $\Pi$  is not isomorphic to the semigroup  $\Pi_1$ , then  $ab \neq \Lambda$  in  $\Pi$ .*

**Proof.** Suppose that  $ab = \Lambda$  in  $\Pi$ . Then every word in the alphabet of the semigroup  $\Pi$  will be equal in  $\Pi$  to some reduced word of the form  $b^\alpha a^\beta$ . The same holds for  $\Pi_1$ . Every equality  $A = B$  holding in  $\Pi_1$  will hold in  $\Pi$ . But, by hypothesis,  $\Pi$  and  $\Pi_1$  are not isomorphic. This means that there exist words  $A_1$  and  $B_1$ , containing only the letters  $a$  and  $b$ , such that  $A_1 = B_1$  in  $\Pi$  and

$A_1 \neq B_2$  in  $\Pi_1$ . Find reduced words equal in  $\Pi_1$  to the words  $A_1$  and  $B_1$ . We obtain  $A_1 = b^\alpha a^\beta$  and  $B_1 = b^\gamma a^\delta$  in  $\Pi_1$ . Since, by hypothesis,  $A_1 \neq B_1$  in  $\Pi_1$ , either  $\alpha \neq \gamma$  or  $\beta \neq \delta$ . Without loss of generality we may assume that  $\alpha > \gamma$ . Then from the equality  $b^\alpha a^\beta = b^\gamma a^\delta$ , multiplying both sides on the left by  $a^\gamma$  and on the right by  $b^\delta$ , we get  $b^{\alpha-\gamma} a^\beta b^\delta = \Lambda$  in  $\Pi$ . Hence it follows that  $b$  is right-invertible, since, by assumption,  $\alpha > \gamma$ . But  $ab = \Lambda$  means the left-invertibility of  $b$ . If  $b$  is invertible in  $\Pi$ , then  $a$  will also be invertible in  $\Pi$ , which is impossible, since  $\Pi$  is not a group. We have obtained a contradiction. Therefore  $ab \neq \Lambda$  in  $\Pi$ .

**Proof of Theorem 1.** Let the special semigroup  $\Pi$  be distinct from  $\Pi_0$  and  $\Pi_1$  and not be a group. We shall prove that no identity relation holds in it.

Suppose first that there are at least 3 letters of the alphabet of the semigroup  $\Pi$  which are not equal to  $\Lambda$  in  $\Pi$ . Then one of three possible cases occurs: 1) one of these letters is invertible in  $\Pi$ ; 2) two of them are not left-invertible in  $\Pi$ ; 3) two of them are not right-invertible in  $\Pi$ .

Since, by the hypothesis of the theorem,  $\Pi$  is not a group, at least one generator of  $\Pi$  is not invertible on the left or on the right. Therefore in case 1) we can apply Lemma 4, and in cases 2) and 3) Lemma 3 is applicable. Thus there remains the case when only two letters of the alphabet  $\mathfrak{A}$  are not equal to  $\Lambda$  in  $\Pi$ . Let these letters be  $a$  and  $b$ . Since, by hypothesis,  $\Pi$  is not isomorphic to  $\Pi_0$  or  $\Pi_1$ , we have  $a \neq \Lambda$ ,  $b \neq \Lambda$ , and  $ab \neq \Lambda$  in  $\Pi$ . The last inequality holds by virtue of Lemma 8. Obviously, we may assume that the left-hand sides of the defining relations of the form (1) of the semigroup  $\Pi$  contain only the letters  $a$  and  $b$ . If there are no such relations at all, then  $\Pi$  is a free semigroup with two generators  $a$  and  $b$ , which will be noninvertible on the left in  $\Pi$ . Then Lemma 3 is applicable. We may assume that in  $\Pi$  there is a relation  $A_1 = \Lambda$ , where  $A_1$  is a nonempty word. Without loss of generality we may suppose that  $A_1$  begins with the letter  $a$ . If, moreover,  $A_1$  also ends with the letter  $a$ , then  $a$  is invertible in  $\Pi$ . But, by hypothesis,  $\Pi$  is not a group. Then  $b$  is not left- or right-invertible in  $\Pi$ , and hence Lemma 4 is applicable. Therefore we may assume that  $A_1$  ends with the letter  $b$ . Then we have  $A_1 \doteq a^{\gamma_1} b A_2$ , where  $\gamma_1 > 0$ .

Two cases are possible:

- I. For every natural  $\gamma$  the word  $a^\gamma b$  is not invertible in  $\Pi$ .
- II. For some  $\gamma$  the word  $a^\gamma b$  is invertible in  $\Pi$ .

In case I Lemma 5 is applicable, since the relation  $A_1 = \Lambda$  means that  $a^{\gamma_1} b$  is right-invertible.

Case II —let  $a^\gamma b$  be invertible in  $\Pi$ . If  $\gamma = 0$ , then  $b$  is invertible in  $\Pi$ . Then  $a$  is not invertible either on the left or on the right. Lemma 4 is applicable here. If  $\gamma = 1$ , then Lemma 7 is applicable, since, by the condition stated above,  $ab \neq \Lambda$  in  $\Pi$ . If, however,  $\gamma > 1$ , then identical relations in  $\Pi$  are impossible by Lemma 6. Theorem 1 is proved.

The semigroups  $\Pi_0$  and  $\Pi_1$  are not groups. The question of identical relations in  $\Pi_0$  is settled trivially. It is commutative.

**Theorem 2.** *In the semigroup  $\Pi_1$  the identity \**

$$XY^2XYX^2Y^2X = XY^2X^2YXY^2X. \quad (4)$$

**Proof.** Since every word of the semigroup  $\Pi_1$  is equal in  $\Pi_1$  to some word of the form  $b^\alpha a^\beta$ , it suffices to verify relation (4) for  $X = b^\alpha a^\beta$  and  $Y = b^\gamma a^\delta$ , for arbitrary  $\alpha, \beta, \gamma, \delta \geq 0$ . Taking into account that  $a^p b^q = a^{p-q}$  for  $q \leq p$  and  $a^p b^q = b^{q-p}$  for  $p \leq q$ , it is not difficult to verify directly that the left- and right-hand sides of relation (4), after the substitution  $X = b^\alpha a^\beta$ ,  $Y = b^\gamma a^\delta$ , will be equal in  $\Pi$ .

It is expedient to consider 8 cases:

- 1)  $\delta \leq \alpha$ ,  $\delta \leq \gamma$ ,  $\beta \leq \gamma$ ; 5)  $\delta > \alpha$ ,  $\delta > \gamma$ ,  $\beta > \gamma$ ;
- 2)  $\delta \leq \alpha$ ,  $\delta \leq \gamma$ ,  $\beta > \gamma$ ; 6)  $\delta > \alpha$ ,  $\delta > \gamma$ ,  $\beta \leq \gamma$ ;
- 3)  $\delta \leq \alpha$ ,  $\delta > \gamma$ ,  $\beta > \gamma$ ; 7)  $\delta > \alpha$ ,  $\delta \leq \gamma$ ,  $\beta \leq \gamma$ ;
- 4)  $\delta \leq \alpha$ ,  $\delta > \gamma$ ,  $\beta \leq \gamma$ ; 8)  $\delta > \alpha$ ,  $\delta \leq \gamma$ ,  $\beta > \gamma$ .

Here cases 2), 3), 6), and 7) split into two subcases corresponding to the inequalities  $\alpha - \delta \leq \beta - \gamma$  and  $\alpha - \delta > \beta - \gamma$ .

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\* The length of the left- and right-hand sides of identity (4) is 10.

It can be shown that in  $\Pi_1$  no identity holds whose left- or right-hand side would have length  $< 10$ .

*Note: Figure translations are in progress. See original paper for figures.*

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