MECHANICS

K. P. STANYUKOVICH

1962

SovietRxiv

Abstract

Full Text

MECHANICS

K. P. STANYUKOVICH

THE LAGRANGIAN IN RELATIVISTIC MECHANICS OF CONTINUOUS MEDIA

(Presented by Academician N. N. Bogolyubov, 19 II 1962)

In the classical mechanics of continuous media, the Lagrangian is established comparatively simply, which makes it possible to write the corresponding quantum conditions, for example, for sound waves and in considering phenomena of superfluidity ([1]). In the relativistic mechanics of continuous media only in particular cases, when the equation of state of the medium has the form $p = (k-1)\varepsilon$ (in the ultrarelativistic case), has it been possible to find the Lagrangian, which is meaningful in the study of multiple particle production ([2,3]). However, it is useful to know the form of the Lagrangian in the general relativistic case for an arbitrary medium. Then, for example, by adding to the Lagrangian of a continuous medium the Lagrangian of interaction with fields, one can more accurately and simply consider, in the most general case, a number of problems of multiple particle production and a number of other problems.

Let us consider potential motions of a continuous medium. In this case we shall have ([4])

$$WU_i = -c\,\partial S/\partial x_i, \tag{1}$$

where $W = (p+\varepsilon)V$ is the heat content, S is the potential. The continuity equation can be written in the form

$$d\ln V/dS = \partial u_k/\partial x_k. \tag{2}$$

Passing to the ordinary components of velocity, we find that

$$Wa_\alpha/\theta c^2 = -\partial S/\partial x_\alpha = -S_\alpha, \qquad W/\theta = \partial S/\partial t = \dot{S}, \tag{3}$$

$$\partial\delta/\partial t + \operatorname{div}(a\delta) = 0 \quad \text{or} \quad \dot{\delta} = -\partial(a_\alpha\delta)/\partial x_\alpha.$$

Here $a_\alpha/c = -cS_\alpha/\dot{S}$, $\delta = 1/V\theta$, $\theta = \sqrt{1-a^2/c^2}$.

Here the only scalar quantity (invariant with respect to Lorentz transformations) is the pressure. Since the Lagrangian L must also be a scalar quantity, one may put $L = \text{const} \cdot p$, and, proceeding from the condition that

$$H = E = -\frac{\partial L}{\partial \dot{S}}\,\dot{S} - L, \tag{4}$$

where $H = E = \varepsilon V$ is the energy (Hamiltonian function), it can be shown that const $= 1$; thus

$$L = p. \tag{5}$$

Since $p = (W - E)/V$, then, squaring $WU_i = c\,\partial S/\partial x_i$, we find that

$$W^2 = -c^2 \left(\frac{\partial S}{\partial x_i}\right)^2 = \dot{S}^2 - c^2 S_\alpha^2, \tag{6}$$

whence

$$W = ic\sqrt{(\partial S/\partial x_i)^2} = \sqrt{-c^2 S_\alpha^2 + \dot{S}^2},$$

$$L = \frac{ic\sqrt{(\partial S/\partial x_i)^2} - E}{V} = \frac{\sqrt{-c^2 S_\alpha^2 + \dot{S}^2} - E}{V}. \tag{7}$$

This is the form taken by the Lagrangian for an ideal continuous medium in the relativistic case.

Let us define the derivatives $\partial L/\partial \dot{S}$ and $\partial L/\partial S_a$:

$$\frac{\partial L}{\partial \dot{S}} = \frac{\dot{S}}{V\sqrt{\dot{S}^2 - c^2 S_a^2}} - \frac{1}{V}\left[\frac{\sqrt{\dot{S}^2 - c^2 S_a^2} - E}{V} + \frac{dE}{dV}\right]\frac{dV}{\partial \dot{S}}.$$

Since $(\sqrt{\dot{S}^2 - c^2 S_a^2} - E)/V = p; \quad p + dE/dV = 0$, we have

$$\partial L/\partial \dot{S} = \dot{S}/WV = 1/\theta V = \delta;$$

$$\partial L/dS_a = -c^2 S_a/V\sqrt{\dot{S}^2 - c^2 S_a^2} = -c^2 S_a/\dot{S}\theta V = a_a/\theta V = \delta a_a.$$

The equation must hold

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{S}} + \frac{\partial}{\partial x_a}\frac{\partial L}{\partial S_a} = 0.$$

Indeed, we have

$$\partial\delta/\partial t + \mathrm{div}(a_a\delta) = 0.$$

Thus, the variables S and δ are canonically conjugate, and

$$L = \frac{\partial L}{\partial \dot{S}}\dot{S} - H = \delta\dot{S} - H. \tag{8}$$

Let us define, in order to check our calculations,

$$T_{ik} = -\frac{\partial S}{\partial x_k}\frac{\partial L}{\partial \partial S/\partial x_i} + \delta_{ik}L = -S_k\frac{\partial L}{\partial S_i} + \delta_{ik}p.$$

It is obvious that

$$\partial L/\partial S_i = icS_i/V\sqrt{S_j^2} = -c^2S_i/WV = cWU_i/VW = cU_i/V,$$

which gives T_{ik} in the form:

$$T_{ik} = WU_iU_k/V + \delta_{ik}p = (p+\varepsilon)U_iU_k + \delta_{ik}p. \tag{9}$$

We have arrived at the correct value of T_{ik}.

Let us now write the continuity equation (2) in the form

$$U_k\,\partial\ln V/\partial x_k = \partial U_k/\partial x_k;$$

since $U_k = -\frac{c}{W}\frac{\partial S}{\partial x_k}$, $\frac{d\ln W}{d\ln V} = -\frac{\omega^2}{c^2}$, we have

$$\frac{1}{W}\frac{\partial S}{\partial x_k}\frac{\partial \ln W}{\partial x_k} = -\frac{\omega^2}{c^2}\frac{\partial}{\partial x_k}\left(\frac{1}{W}\frac{\partial S}{\partial x_k}\right) = \frac{\omega^2}{Wc^2}\left[\frac{\partial\ln W}{\partial x_k}\frac{\partial S}{\partial x_k} - \frac{\partial^2 S}{\partial x_k^2}\right]$$

or

$$\left(1-\frac{\omega^2}{c^2}\right)\frac{\partial \ln W}{\partial x_k}\frac{\partial S}{\partial x_k} + \frac{\omega^2}{c^2}\frac{\partial^2 S}{\partial x_k^2} = 0. \tag{10}$$

This equation and equation (6), for a given state $W = W(V)$, completely describe the relativistic irrotational motions of an ideal continuous medium.

From (6) it follows that

$$\frac{\partial \ln W}{\partial x_k} = \frac{\partial^2 S}{\partial x_i \partial x_k} \frac{\partial S}{\partial x_i} / (\partial S / \partial x_i)^2,$$

and therefore (10) may be written in the form

$$\left(1 - \frac{\omega^2}{c^2}\right) \frac{\partial S}{\partial x_k} \frac{\partial S}{\partial x_i} \frac{\partial^2 S}{\partial x_k \partial x_i} = \frac{1}{2} \left(1 - \frac{\omega^2}{c^2}\right) \frac{\partial S}{\partial x_k} \frac{\partial}{\partial x_k} \left(\frac{\partial S}{\partial x_i}\right)^2 = -\frac{\omega^2}{c^2} \left(\frac{\partial S}{\partial x_i}\right)^2 \frac{\partial^2 S}{\partial x_i^2}, \tag{11}$$

or, changing the index i to k:

$$\frac{1}{2} \left(\frac{c^2}{\omega^2} - 1\right) \frac{\partial S}{\partial x_k} \frac{\partial}{\partial x_k} \left(\frac{\partial S}{\partial x_k}\right)^2 + \left(\frac{\partial S}{\partial x_k}\right)^2 \frac{\partial^2 S}{\partial x_k^2} = 0. \tag{12}$$

Equation (12) can be written in the form

$$-\frac{\omega^2}{c^2} W \frac{\partial S_k}{\partial x_k} + \left(1 - \frac{\omega^2}{c^2}\right) S_k \frac{\partial W}{\partial x_k} = 0. \tag{13}$$

Using the isentrope equation

$$p = AV^{-k}, \tag{14}$$

we find that

$$W = \alpha c^2 + \frac{k}{k-1} AV^{1-k}, \tag{15}$$

where $\alpha = 1$ for relativistic motions of the medium and $\alpha = 0$ for ultrarelativistic motions. In this case

$$\frac{\omega^2}{c^2} = -\frac{d \ln W}{d \ln V} = (k-1)\left(1 - \frac{\alpha c^2}{W}\right); \qquad \frac{c^2}{\omega^2} - 1 = \frac{(2-k)/(k-1) + \alpha c^2/W}{1 - \alpha c^2/W},$$

and equation (13) takes the form

$$\left(\frac{2-k}{k-1} W + \alpha c^2\right) S_k \frac{\partial W}{\partial x_k} + (W - \alpha c^2) W \frac{\partial S_k}{\partial x_k} = 0, \tag{16}$$

or

$$\left(\frac{2-k}{k-1}\sqrt{-S_i^2}+\alpha c\right)S_k S_i S_{ik}+\left(\sqrt{-S_i^2}-\alpha c\right)S_i^2 S_{kk}=0. \tag{17}$$

This equation can be represented in the form:

$$\frac{\partial}{\partial x_k}\left[S_k(S_i^2)^{(2-k)/2(k-1)}\right]=\frac{\alpha c}{\sqrt{-S_i^2}}\left(S_i^2 S_{kk}-S_i S_k S_{ik}\right)=\alpha c S_i^2\frac{\partial}{\partial x_k}\frac{S_k}{\sqrt{-S_i^2}}. \tag{18}$$

In the case $\alpha=0$ we have

$$\frac{\partial}{\partial x_k}\left[S_k(S_i^2)^{(2-k)/2(k-1)}\right]=0; \tag{19}$$

if $k=4/3$—the limiting ultrarelativistic case—then

$$\frac{\partial}{\partial x_k}\left[S_k S_i^2\right]=0, \tag{20}$$

and we arrive at the equation previously obtained by I. M. Khalatnikov ([5]). Equations (18)—(20) are very convenient for investigation by the method of characteristics. Equations (19), (20) reduce, as is known, to the Riemann equation ([5],[6]).

Let us now consider purely ultrarelativistic flows. For this purpose we write the basic equations of relativistic hydrodynamics in the form ([7])

$$\frac{dU_i}{dS}(p+\varepsilon)+U_i\frac{dp}{dS}+\frac{\partial p}{\partial x_i}=0; \tag{21}$$

$$\frac{d\varepsilon}{dS}+(p+\varepsilon)\frac{\partial U_k}{\partial x_k}=0. \tag{22}$$

Let $p=(k-1)\varepsilon$; then equation (21) can be reduced to the form

$$\frac{d}{dS}\left(p^{(k-1)/k}U_i\right)+\frac{\partial p^{(k-1)/k}}{\partial x_i}=0. \tag{23}$$

This equation is equivalent to the equation

$$\partial\left(p^{(k-1)/k}U_i\right)/\partial x_k-\partial\left(p^{(k-1)/k}U_k\right)/\partial x_i=0. \tag{24}$$

Consequently, in the ultrarelativistic case not only isentropic but also adiabatic flows have a potential, so that

$$c\,\partial S/\partial x_i = -c^2 U_i\,(p/p_0)^{(k-1)/k}, \tag{25}$$

where p_0 is some initial pressure. This equation is completely analogous to equation (1), if one replaces $W \to c^2(p/p_0)^{(k-1)/k}$. Equation (22) then takes the form

$$\partial p/\partial S + kp\,\partial U_k/\partial x_k = 0. \tag{26}$$

This equation is analogous to equation (3), if one replaces $\delta \to \dfrac{1}{V_0\theta}\left(\dfrac{p}{p_0}\right)^{1/k}$, where V_0 is the initial value of V. Consequently, all the remaining relations and equations under these substitutions will retain their form also for adiabatic ultrarelativistic flows of the medium.

In conclusion, it is useful to write the Lagrangian (7) in a somewhat different form. Since

$$W = \alpha c^2 + \frac{k}{k-1} A^{1/k} p^{(k-1)/k} = ic\,(S_i^2)^{1/2},$$

then

$$L = p = B\left[(-S_i^2)^{1/2} - \alpha c\right]^{k/(k-1)}, \tag{27}$$

where $B = \left(\dfrac{k-1}{k}c\right)^{k/(k-1)} A^{-1/(k-1)}$. In this case the field equation

$$\frac{\partial L}{\partial S} - \frac{\partial}{\partial x_i}\frac{\partial L}{\partial S_i} = 0 \tag{28}$$

immediately gives equations (17).

In the particular case of ultrarelativistic motions,

$$L = B(-S_i^2)^{k/2(k-1)}; \tag{29}$$

for $k = 4/3$,

$$L = B(-S_i^2)^2, \tag{30}$$

where $B = (c/4)^4 A^{-3} = (c/4p_0V_0)^4 p_0$.

It is easy to find also expressions for the Lagrangian in the classical case. Let $S = S_{\rm cl} + c^2 t$, $W = c^2 + W_{\rm cl}$; then $S_\alpha = -a_\alpha$, $\dot{S} = W_{\rm cl} + a^2/2 = W_{\rm cl} + S_\alpha^2/2$. Since $p = \delta(W_{\rm cl} - E_{\rm cl})$, we have

$$L = \delta\,[\dot{S} - S_\alpha^2/2 - E_{\rm cl}], \tag{31}$$

and we arrive at the known expression for the Lagrangian (1).

In conclusion, I express my gratitude to S. M. Kolesnikov and M. I. Kiselev for discussing the results of the work.

All-Union Scientific Research
Institute of Electromechanics

Received
17 II 1962

CITED LITERATURE

1. L. D. Landau, ZhETF, **5**, 71 (1941).
2. G. A. Milekhin, Reports at the Kiev Conference on High-Energy Physics, 1959, 1960.
3. W. Heisenberg, *Nonlinear Quantum Field Theory*, IL, 1959.
4. F. A. Baum, S. A. Kaplan, K. P. Stanyukovich, *Introduction to Cosmic Gas Dynamics*, part 3, § 2, 1958.
5. I. M. Khalatnikov, ZhETF, **27**, issue 5, 529 (1954).
6. K. P. Stanyukovich, DAN, **139**, No. 3 (1961).
7. K. P. Stanyukovich, *Unsteady Motions of a Continuous Medium*, § 85, 1955.

Note: Figure translations are in progress. See original paper for figures.