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Abstract

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INVESTIGATION OF SOLUTIONS OF THE BASIC EQUATIONS OF THE THEORY OF SURFACES

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In the present note we study solutions $w(z)$ of the Gauss and Codazzi equations for surfaces of positive curvature. Using the specificity of the equations, a new integral representation for $w(z)$ is proved. We also derive a general formula for solutions of the Gauss and Codazzi equations and show that there exists a one-to-one correspondence between the functions $w(z)$ and analytic functions. Next, for the system of the Gauss and Codazzi equations, a nonlinear boundary-value problem is considered, the exact formulation of which will be given below. With the aid of the formula proved, the problem is reduced to finding an analytic function. Using the results of F. D. Gakhov ([1]), the unsolvability of the posed nonlinear boundary-value problem is proved for the linear Hilbert problem. An integral representation, somewhat different from the one given here, was obtained by K. M. Belov ([2]).

1. We consider simply connected surfaces of strictly positive curvature K up to the boundary, belonging to the class $D_{3,p}(D)$, $p > 2$ (the radius vector $\mathbf{r}(u,v)$ has three generalized Sobolev derivatives summable with exponent p). The boundary of the surface is assumed to be a piecewise smooth simple closed curve, consisting of a finite number of arcs of class C^1_μ, $0 < \mu \leq 1$. The interior angles at the corner points of this curve are different from zero. Without loss of generality, we assume the domain D of variation of the parameters u, v to be the unit disk with boundary Γ.

2. Let a surface S with boundary be given. Introduce on it an isometrically conjugate parametrization u, v, in which the second quadratic form is represented in the form

$$II = L_0(u,v)\,(du^2 + dv^2),$$

where $L_0 > 0,\ L_0 \in D_{1,p}(D),\ p > 2.$

Let the surface S be isometrically transformed into a surface S^* with coefficients L, M, N of the second fundamental form. Following the general idea of I. N.

Vekua ([3]), we write the Gauss and Codazzi equations for the surface S^* in the following form (see also our note ([5])):

$$\partial_{\bar{z}} w + A_1(z, w, \bar{w}_z)\, w + B_1(z, w, \bar{w}_z)\, \bar{w} = 0, \tag{1}$$

where

$$A_1 = A - \frac{C\bar{w}}{L_0 + \sqrt{L_1^2 + w\bar{w}}} + \frac{i\partial_z L_0 w}{(L_0 + \sqrt{L_0^2 + w\bar{w}})\sqrt{L_0^2 + w\bar{w}}} - \frac{i\partial_z w}{2\sqrt{L_0^2 + w\bar{w}}};$$

$$B_1 = B - \frac{i\partial_z w}{2\sqrt{L_0^2 + w\bar{w}}}; \tag{2}$$

here $w = M + \dfrac{i}{2}(L - N)$ is the unknown function; the coefficients A, B, C are expressed in a known manner in terms of the Christoffel symbols and belong to the class $C_{\frac{p-2}{2}}(D + \Gamma)$, $p > 2$.

The solution of equation (1) is sought in the class $D_{1,p}(D)$, $p > 2$.

It follows from (2) that $A_1, B_1 \in L_p(D + \Gamma)$, $p > 2$, if $w \in D_{1,p}(D)$, $p > 2$. Consequently, according to (3), the following is valid.

Lemma 1. Let $w(z)$ be a solution of equation (1). Let

$$g(z) = \begin{cases} A_1 + B_1 \dfrac{\overline{w}}{w}, & \text{if } w(z) \neq 0,\ z \in D, \\ A_1 + B_1, & \text{if } w(z) = 0,\ z \in D. \end{cases}$$

In that case the function

$$\varphi(z) = w(z) e^{-\omega(z)}, \tag{3}$$

where

$$\omega(z) = \frac{1}{\pi} \iint_D \frac{g(\zeta)\, d\xi d\eta}{\zeta - z} \equiv -Tg,$$

is analytic in D.

Thus, by formula (3), to every solution of equation (1) there corresponds a unique analytic function. We shall prove the converse.

Lemma 2. Every solution of equation (1) can be represented in a unique way in the form

$$w(z) = \varphi(z)\left[1 - \frac{1}{\pi}\iint_D \frac{\rho(\zeta,\varphi(\zeta))\,d\xi d\eta}{\zeta - z}\right], \tag{4}$$

where $\varphi(z)$ is analytic in D; $\rho(z,\varphi(z))$ is a certain nonlinear operator, $\rho \in L_p(D+\Gamma)$, $p > 2$.

Proof. We shall show the one-to-one invertibility of the integral representation (3) with respect to the function $w(z)$. Following (1), we shall seek the function $e^{\omega(z)} = V(z)$, from which we must require that it be: 1) continuous in the entire plane; 2) analytic in D^- (the complement of $D+\Gamma$ to the full plane) and satisfy there the condition $V(\infty) = 1$; 3) satisfy in D the equation

$$\partial_{\bar z} V + \mu_1(z,\varphi,V)\partial_z V + \mu_2(z,\varphi,V)\partial_{\bar z}\overline{V} + d(z,\varphi,V) = 0, \tag{5}$$

where $|\mu_1| = |\mu_2| \le \mu_0 < 1$ uniformly with respect to φ, V for $z \in D$; $d = d_0(z,\varphi,V) + d_1(z,\varphi,V)V + d_2(z,\varphi,V)\overline{V}$, $\|d_i(z,\varphi,V)\|_{L_p} < K_1$ uniformly with respect to φ, V for $z \in D$.

Equation (5) is obtained by differentiating (3) under conditions (1), (2). We shall seek a solution $V(z)$ of equation (5) in the form

$$V(z) = \varphi_0(z) - \frac{1}{\pi}\iint_D \frac{\rho(\zeta)\,d\xi d\eta}{\zeta - z}, \tag{6}$$

where $\varphi_0(z)$ is analytic in D, $\rho = 0$ outside D. It is known [3] that representation (6), for a given $\varphi_0(z)$, is unique. By virtue of properties 1), 2), 3) of the function $V(z)$, it is easy to establish that $\varphi_0(z) \equiv 1$. Substituting (6) into (5), we obtain for $\rho(z)$ the equation

$$\rho + \mu_1(z,\varphi,1+T\rho)S\rho + \mu_2(z,\varphi,1+T\rho)\overline{S\rho} + d(z,\varphi,1+T\rho) = 0, \tag{7}$$

where $S\rho = \dfrac{\partial}{\partial z}T\rho$.

As shown in [4], the integral equation (7), under the conditions indicated above, is always solvable for a given function $\varphi(z)$. The solution $\rho \in L_p(D+\Gamma)$, $p > 2$, which proves the lemma.

3. Consider for equation (1) the following nonlinear boundary-value problem:

$$\mathrm{Re}\{\overline{\lambda(t)}\,w(t)\} + \Phi(w;t) = 0, \qquad t \in \Gamma, \tag{8}$$

where $\lambda(t)$ and $\Phi(w;t)$ are given functions, Hölder-continuous. We require that they satisfy the following conditions: 1) the index

$$n = \frac{1}{2\pi}\Delta_\Gamma \arg\lambda(t) < 0$$

($\Delta_\Gamma f(t)$ is the increment of the function $f(t)$ when the point t describes the curve Γ once in the direction leaving the domain D on the left), $|\lambda(t)| = 1$; 2) the function $\Phi(w;t) \geq 0$ (or $0 \leq 1$) for $t \in \Gamma$ and arbitrary w.

Theorem. The boundary-value problem (1), (8) has no solutions distinct from zero.

Proof. Substituting the expression $w(z)$ from formula (3) into the boundary condition (8) and denoting $\overline{\lambda(t)}e^{\omega(t)} = \lambda_0(t)$, we arrive at the boundary-value problem

$$\mathrm{Re}\{\overline{\lambda_0(t)}\varphi(t)\} + \Phi(\varphi e^{\omega};t) = 0 \tag{9}$$

for an analytic function. Although the function $\lambda_0(t)$ itself is unknown to us, we can compute its index exactly ([1]):

$$\frac{1}{2\pi}\Delta_\Gamma \arg\lambda_0(t) = n < 0.$$

By the methods of work ([1]), the boundary-value problem (9) can be reduced to $2|n|$ integral equations with respect to the function $\varphi(z)$. One of these equations has the following form:

$$\int_0^{2\pi} |\lambda_0(s)|^{-1} e^{q(s)}\Phi(\varphi e^{\omega};s)\,ds = 0, \tag{10}$$

where the real function $q(s) \in C_\alpha(\Gamma)$, $0 < \alpha \leq 1$.

Since, by assumption, the function $\Phi(\varphi e^{\omega};t)$ does not change sign on the contour, equation (10) has a solution only when $\Phi \equiv 0$. But then $\varphi \equiv 0$ in D, i.e. $w \equiv 0$.

4. From the integral representation (3) of the solutions of the Gauss and Codazzi equations it follows, for example, that on nontrivially isometric surfaces the points of congruence lie isolated. Lemma 2 makes it possible to prove the following proposition: every surface of positive curvature with boundary admits such continuous bendings that any m fixed points remain points of congruence under them.

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Note: Figure translations are in progress. See original paper for figures.