RINGS OVER WHICH FLAT MODULES ARE FREE

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Abstract

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MATHEMATICS

V. E. GOVOROV

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In Bass' s paper ([2]) it was established that the direct limit of a direct spectrum of projective Λ-modules is a projective Λ-module if and only if Λ is perfect. In this case flat modules are projective. In the present paper it is shown that flat modules over an arbitrary ring are the direct limit of a direct spectrum of free modules. Hence it follows that the problem: over which rings are flat modules free (projective), is equivalent to the problem: over which rings is the direct limit of a direct spectrum of free (projective) modules a free (projective) module. It is further proved that, in order that all left flat Λ-modules be free, it is necessary and sufficient that Λ be a local ring with left T-nilpotent maximal ideal.

Let Λ be an arbitrary associative ring with identity. All modules considered will be assumed to be unitary.

Lemma 1. *A left Λ-module A is flat if and only if it is the direct limit of a direct spectrum of projectives.*

That the direct limit of a direct spectrum of projective modules is flat follows from the commutativity of the functor $\mathrm{Tor}_1^\Lambda(A, B)$ with the operation of taking the direct limit.

Now let A be a flat left Λ-module; then there exists a free module F and an exact sequence

$$0 \to A' \to F \to A \to 0. \tag{1}$$

The module A' is pure (see ([1])) in F. Indeed, sequence (1) induces, for an arbitrary right Λ-module B, the exact sequence

$$0 \to \mathrm{Tor}_1^\Lambda(B, A) \to B \otimes_\Lambda A' \to B \otimes_\Lambda F \to B \otimes_\Lambda A \to 0;$$

but $\mathrm{Tor}_1^\Lambda(B, A) = 0$, since A is flat; consequently, A' is pure in F.

Take an arbitrary finite set of elements $a_1, a_2, \dots, a_n \in A'$, and let $\{u_\alpha\}$ be a system of free generators of the module F. Then

$$a_j = \sum_{i=1}^{s_j} \lambda_{ji} u_i \quad (j = 1, 2, \dots, n).$$

By Theorem 2.4 of ([1]) there exist elements

$$u_1', u_2', \dots, u_s' \in A'$$

such that

$$a_j = \sum_{i=1}^{s_j} \lambda_{ji} u_i',$$

and, consequently, there exists an epimorphism φ of the module F onto the submodule C' of the module A', generated by the elements $u_1', u_2', \dots, u_s'$, defined by the equalities: $\varphi(u_i) = u_i'$ $(i = 1, 2, \dots, s)$, $\varphi(u_k) = 0$ $(k \neq 1, 2, \dots, s)$, i.e. C' is a direct summand in F. Thus, for every finite subset of elements of A' one can find a projective submodule C' of the module A' containing this subset; that is, the module A' is the direct limit of a direct spectrum of projective modules.

$\{C_\alpha\}$; but then the factor modules F/C_α are projective and form a direct spectrum with respect to the ordering $F/C_\alpha < F/C_\beta$, if $C_\alpha \subset C_\beta$, and the natural homomorphisms. The limit of the spectrum $\{F/C_\alpha\}$ is the module $F/A' \simeq A$ (see ([4]), p. 284).

Lemma 2. *A Λ-module A is flat if and only if it is the limit of a direct spectrum of free Λ-modules.*

Every flat Λ-module A can, by Lemma 1, be represented as the limit of a direct spectrum $\{\pi_\alpha^\beta, P_\alpha\}$ of projective Λ-modules P_α. If for some free module F there is a decomposition $F = P_\alpha \oplus Q_\alpha$, then the module P_α can be represented as the limit of a direct spectrum of free Λ-modules

$$F \xrightarrow{\varphi_1} F \xrightarrow{\varphi_2} \cdots,$$

where $\operatorname{Ker} \varphi_i = Q_\alpha$. Let now $\{\pi_\alpha^\beta, P_\alpha\}$ be a direct spectrum of projective Λ-modules and $\{q_{\alpha k}^{k+1}, F_{\alpha,k}\}$ a direct spectrum of free Λ-modules constructed as indicated above. The set of all $F_{\alpha k}$ is naturally ordered. We define the homomorphisms $\varphi_{\alpha k}^{\beta l} : F_{\alpha k} \to F_{\beta l}$ as follows:

$$\varphi_{\alpha k}^{\beta l} = i_{\beta l} \pi_\alpha^\beta \varphi_{\alpha k},$$

where $i_{\beta l}$ is a monomorphism of P_β into $F_{\beta l}$ such that $\varphi_{\beta l} i_{\beta l}$ is the identity mapping. With respect to the homomorphisms $\varphi_{\alpha k}^{\beta l}$, the set $F_{\alpha k}$ forms a direct spectrum; indeed:

$$\varphi_{\beta l}^{\gamma s} \varphi_{\alpha k}^{\beta l} = i_{\gamma s} \pi_\beta^\gamma \varphi_{\beta l} i_{\beta l} \pi_\alpha^\beta \varphi_{\alpha k} = i_{\gamma s} \pi_\beta^\gamma \pi_\alpha^\beta \varphi_{\alpha k} = i_{\gamma s} \pi_\alpha^\gamma \varphi_{\alpha k} = \varphi_{\alpha k}^{\gamma s}.$$

It is obvious that the limit of $\{\varphi_{\alpha k}^{\beta l}, F_{\alpha k}\}$ is the same as that of the spectrum $\{\pi_\alpha^\beta, P_\alpha\}$.

Theorem 1. *In order that all flat left Λ-modules be free, it is necessary and sufficient that Λ be a local ring with a T-nilpotent* ([2]) *maximal ideal on the left.*

Lemma 3. *If all left flat Λ-modules are free, then every element $\lambda \in \Lambda$ is either invertible or is a zero divisor.*

Proof. Suppose $\lambda \in \Lambda$ is neither invertible nor a zero divisor; then Λ, as a left Λ-module, has a proper submodule $\Lambda\lambda$, isomorphic to the Λ-module Λ. This permits one to construct the following direct spectrum of modules isomorphic to Λ:

$$\Lambda_1 \xrightarrow{\varphi_1} \Lambda_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} \Lambda_n \xrightarrow{\varphi_n} \cdots,$$

where φ_n is a monomorphism of Λ onto $\Lambda\lambda$.

By assumption, the limit of this spectrum is a free Λ-module F with an infinite basis, since if the basis were finite, there would be a number n such that all elements of the basis would be contained in Λ_n, but Λ_n is a proper submodule of the module Λ_{n+1}.

Let $w_1, w_2, \dots, w_n, \dots$ be a basis of F, and let $a_1, a_2, \dots, a_n, \dots$ be free generators of the modules $\Lambda_1, \Lambda_2, \dots, \Lambda_n, \dots$, respectively. Take w_1 and w_2; for them there is a number n and elements $\mu, \nu \in \Lambda$ such that $\mu a_n = w_1,\ \nu a_n = w_2$. On the other hand, a_n lies in F and, therefore, $a_n = \sum \lambda_i w_i$; hence we obtain

$$\mu\lambda_1 = 1, \qquad \mu\lambda_2 = 0, \qquad \nu\lambda_1 = 0, \qquad \nu\lambda_2 = 1.$$

But then $\lambda_1\mu$ and $\lambda_2\mu$ are orthogonal idempotents not equal to zero; if a ring has an idempotent different from zero and one, then the ring decomposes into a direct sum of left ideals, i.e. there exists a projective but not free Λ-module. The lemma is proved.

From the absence of nontrivial idempotents it follows that a left inverse coincides with a right inverse; indeed, if $ab = 1$, then ba is an idempotent not equal to zero and, consequently, $ba = 1$.

Lemma 4. *The noninvertible elements in the ring Λ form a T-nilpotent ideal.*

First of all, let us prove the T-nilpotence of the set of all noninvertible elements, i.e., if $a_1, a_2, \dots, a_n, \dots$ are noninvertible, then there exists a natural number s such that $a_1 a_2 \dots a_s = 0$. Take a free left Λ-module F with basis $x_1, x_2, \dots, x_n, \dots$, and in it the submodule G generated by the elements $z_i = x_i - a_{ix_{i+1}}$ $(i = 1, 2, \dots)$. The module G is free. By G_n we shall denote the submodule of the module G generated by $z_1, z_2, \dots, z_n$. Then F/G_n is a free Λ-module. The module F/G is the limit of a direct spectrum of free Λ-modules and, consequently, is free by hypothesis. Denote by $t_1, t_2, \dots$ the images of the elements $x_1, x_2, \dots$ under the mapping of F onto F/G. Suppose that F/G is a free Λ-module with free generators $v_1, v_2, \dots$. Then

$$v_1 = \sum \lambda_i t_i,$$

and since $t_i = a_i t_{i+1}$, there exists a k such that

$$v_1 = \sum_i \lambda_i a_i \dots a_k t_{k+1},$$

where

$$t_{k+1} = \sum_i \mu_j v_j - \mu_1 v_1, \quad v_1 = \left(\sum_i \lambda_i a_i \dots a_k \right) \left(\sum \mu_j v_j + \mu_1 v_1 \right), \quad 1 = \sum_i \lambda_i a_i \dots a_k \mu_1,$$

and, since left inverses are right inverses, it follows that

$$1 = \mu_1 \sum \lambda_i a_i \dots a_k,$$

$$\mu_1 v_1 = t_{k+1} = a_{k+1} t_{k+2}, \quad \text{but} \quad t_{k+2} = \sum \nu_j v_j, \quad \mu_1 v_1 = \sum a_{k+1} \nu_j v_j, \quad \mu_1 = a_{k+1} \nu_1,$$

i.e., we have obtained that a_{k+1} is invertible, contrary to the assumption. Thus it has been proved that all noninvertible elements in the ring Λ form a T-nilpotent ideal. In particular, every noninvertible element a of the ring Λ is nilpotent and, consequently, $1 - a$ is invertible, and all elements that have no inverses form an ideal which is obviously maximal and T-nilpotent. Lemma 4 is proved.

Suppose now that Λ is a local ring with a maximal T-nilpotent ideal. Then Λ is perfect (see ([2])) and, consequently, all flat Λ-modules are projective, and since Λ is local, all projective modules are free (see ([3])). Theorem 1 is proved.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.